CHAPTER 1: PHYSICAL QUANTITIES AND UNITS

DIMENSIONS OF A PHYSICAL QUANTITY

Physical quantities are divided into two groups:

(i) **Fundamental quantities**: are quantities which cannot be expressed in terms of any other quantities using any mathematical equation. Examples of fundamental physical quantities are; mass (M), Length (L) and time (T).

(ii) **Derived quantities**: are quantities which can be expressed in terms of the fundamental physical quantities of mass, length and time. Derived quantities are also called dimensional quantities OR
Derived quantities are those physical quantities which cannot be obtained by combining the fundamental quantities mathematically. Examples include; acceleration, speed, power, force, etc.
Physical quantities that are not related to mass, length and time are called dimensionless or non-dimensional quantities. Examples include; relative density, refractive index, logarithms, trigonometric ratios, mechanical advantage, pure numbers, etc.

Definition: Dimensions of a physical quantity shows the way a physical quantity (derived quantity is related to the fundamental physical quantities, of mass, length and time
The symbol for dimensions is; [ ]

Examples:

(i) $[mass] = M$

(ii) $[length] = L$

(iii) $[time] = T$

(iv) Area = length × width
$\Rightarrow [Area] = [length] \times [width]$
$= L \times L$
$= L^2$

(v) $velocity = \frac{displacement}{time}$
$\Rightarrow [velocity] = \frac{[displacement]}{[time]}$
$= \frac{L}{T} = LT^{-1}$

(vi) $volume = length \times width \times height$
$\Rightarrow [volume] = [length] \times [width] \times [height]$
$= L \times L \times L$
$= L^3$

(vii) Acceleration $= \frac{change\ in\ velocity}{ime}$

$\Rightarrow$ [Acceleration]$= \frac{[change\ in\ velocity]}{[ime]}$

$= \frac{LT^{-1}}{T} = LT^{-2}$

(viii) $Force = mass\ \times acceleration$

$\Rightarrow [Force] = [mass] \times [acceleration]$

$= M \times LT^{-2}$

$= MLT^{-2}$

(ix) $Workdone = Force\ \times distance$

$\Rightarrow [Workdone] = [Force] \times [distance]$

$= MLT^{-2} \times L = ML^2T^{-2}$

(x) $Power = \frac{workdone}{time}$

$\Rightarrow [Power] = \frac{[workdone]}{[time]}$

$= \frac{ML^2T^{-2}}{T} = ML^2T^{-3}$

Trial question: Find the dimensions for energy, pressure, impulse, momentum

Uses of dimensions

- Checking the consistency of equations
- To derive equations

Checking equations

For a correct equation, the units on the left hand side (LHS) should balance with those on the right hand side (RHS). This implies that the dimensions on the LHS should balance with those on the RHS. All correct equations should be dimensionally consistent. However not all dimensionally consistent equations are correct.

Example

Test the following equations for dimensional consistency;

(i) $S = ut + \frac{1}{2}at^2$

Solution

$[LHS] = L$

$[RHS] = (LT^{-1} \times T) + (LT^{-2} \times T^{-2}) = L + L$

Since dimensions on the RHS = dimensions on the LHS, the equation is dimensionally consistent.

(ii) $v^2 = 8u^2 + 4as$

Solution

$[LHS] = (LT^{-1})^2 = L^2T^{-2}$

$[RHS] = L^2T^{-2} + L^2T^{-2}$

Therefore, the dimensions on the RHS are consistent with those on the left hand side. However, it can be seen that the equation is not correct because of the wrong values for the constants 8 and 4. This is due to the fact that the constants 8 and 4 are dimensionless. Therefore, dimensional analysis can be used to eliminate wrong equations, but cannot be used to prove the correctness of a particular equation, since it does not provide any information about the correctness of numerical factors such as $\frac{1}{2}, 2, \pi, e.t.c$

(iii) $V = \sqrt{\frac{Tl}{M}}$ T is the tension in the string of length *l* and mass m where V is the velocity

Solution

$[LHS] = LT^{-1}$

$$[RHS] = \sqrt{\frac{(MLT^{-2}) \times L}{M}}$$

Since [LHS] = [RHS], the equation is dimensionally consistent

(iv) $v = u^2 + at$

Solution

$[LHS] = LT^{-1}$

$[RHS] = L^2T^{-2} + LT^{-1}$

$\therefore [LHS] \neq [RHS] \Rightarrow \text{the equation is wrong}$

Derivation of equations

If one has an idea about the factors upon which a given quantity depends, the method of dimensional analysis can be used to derive equations.

Examples

1. Consider the oscillation of a simple pendulum. The period T may depend on mass, m, length, *l,* and the acceleration due to gravity, g.
   $\therefore T \propto l^x m^y g^z \quad \Rightarrow T = k(l^x m^y g^z)$
   Where k is a dimensionless constant of proportionality

Using dimensions;

$$[LHS] = T$$
$$[RHS] = (L)^x (M)^y (LT^{-2})^z$$
$$\Rightarrow T = (L)^x (M)^y (LT^{-2})^z$$

Equating the corresponding indices of M, L, and T gives:

For M,

0 = y $\quad \therefore y = 0$

For L,

$0 = x + z \quad \therefore \ z = -x$

For T,

$1 = -2z \quad \Rightarrow z = -\frac{1}{2}$

Thus $x = \frac{1}{2}$

By substitution, $T = k\left(l^{\frac{1}{2}} m^0 g^{-\frac{1}{2}}\right)$

$\Rightarrow T = k\sqrt{\frac{l}{g}}$

Note: the method of dimensional analysis cannot give the values of the constant k. However other mathematical methods show that k = 2π

2. The period of vibration of a liquid drop is given by $T = ka^x \rho^y \gamma^z$, where k is a dimensionless constant, a is the radius of the drop, ρ is the density of the liquid and γ is the surface tension whose dimensions are MT^{-2}. $Use\ dimensional$ analysis to find the values of $x, y\ and\ z$.

Solution

$T = ka^x \rho^y \gamma^z$

$[T] = T, [a] = L,\ [\rho] = ML^{-3}\quad, [\gamma] = MT^{-2}\ \Rightarrow T = (L)^x \times (ML^{-3})^y \times (MT^{-2})^z$

Equating the corresponding indices;

For L: $0 = x - 3y \ldots\ldots\ldots (i)$

For M: $0 = y + z \ldots\ldots\ldots\ldots (ii)$

For T: $1 = -2z \ldots\ldots\ldots\ldots.. (iii)$

$\therefore z = -\frac{1}{2}, x = \frac{3}{2}\ , y = \frac{1}{2}$

$\Rightarrow T = ka^{\frac{3}{2}} \rho^{\frac{1}{2}} \gamma^{-\frac{1}{2}} = k\sqrt{\frac{a^3 \rho}{\gamma}}$

3. The volume per second of a liquid flowing through a horizontal pipe of length *l* ,is given by $\frac{k\rho a^x}{\eta l}$, where k is a constant, ρ the excess pressure, a the radius of the pipe and η, the coefficient of viscosity of dimensions $ML^{-1}T^{-1}$. Find *x*

Solution

Volume/second = $\frac{volume}{time} \Rightarrow$ [Volume/second] $= \frac{[volume]}{[time]} = L^3T^{-1}$

$[\rho] = L^{-1}MT^{-2}\ ,\ [a] = L\ ,\ [\eta] = ML^{-1}T^{-1}, [l] = L$

$\therefore L^3T^{-1} = \frac{(L^{-1}MT^{-2})\times(L)^x}{(ML^{-1}T^{-1})\times(L)}$

Equating corresponding indices gives:

$3 = -1 + x$

$\Rightarrow x = 4$

Trial questions:

1. In the formula $T = kE^x P^y \rho^z$, E is the energy, T is the period, P is the pressure, ρ is density and k is a dimensionless constant. Find $x, y\ and\ z$

[Ans: $= \frac{1}{3}, y = -\frac{5}{6}, z = \frac{1}{2}$]

2. For streamline flow of a non-viscous, incompressible fluid, the pressure P at any point is related to the height h and the velocity v by the equation:
   $(P - a) = \rho g(h - b) + \frac{1}{2}\rho(v^2 - d)$, where a, b and d are constants, ρ is the density of the fluid and g is the acceleration due to gravity. Given that the equation is dimensionally consistent, find the dimensions of a, b and d; and hence their units.
   [Ans: $[a] = ML^{-1}T^{-2}$, $[b] = L,\ [d] = L^2T^{-2}$]
3. Assuming that the mass m of the largest stone that can be moved by a flowing river depends on the velocity v of the water, its density ρ and acceleration due to gravity g, show that: $m = \frac{k\rho v^6}{g^3}$ where k is a dimensionless constant.
4. Experiments show that the frequency, f of a tuning fork depends on the length, *l*, of the prong, the density, ρ and young's modulus, Y. Using dimensional analysis, show that
   $f = \frac{k}{l}\left(\frac{Y}{\rho}\right)^{\frac{1}{2}}$ where k is a dimensionless constant
5. Vander Waals equation is given by: $\left(P + \frac{a}{v^2}\right)(v - b) = RT$, where P is the pressure exerted by a gas, T is the absolute temperature, v is volume of a gas, and a and b are constants. Find the dimensions and units of a and b.

CHAPTER 2: KINEMATICS

LINEAR MOTION

Linear motion refers to motion in a straight line

Definitions:

Displacement: This is the distance covered by a body moving in a specified direction. The SI unit of displacement is metres (m).

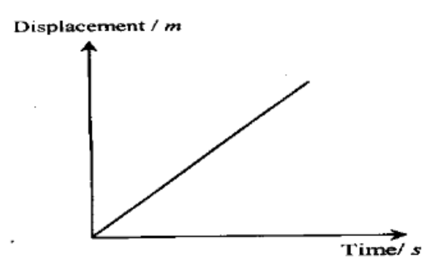
Velocity: this is the rate of change of displacement. The SI unit of velocity is metres per second (ms^{-1})

Uniform velocity: this happens if a body undergoes equal displacements with in equal successive time intervals, no matter how small the time intervals may be.
If the displacements in equal successive time intervals are not equal, the motion is said to be non-uniform.

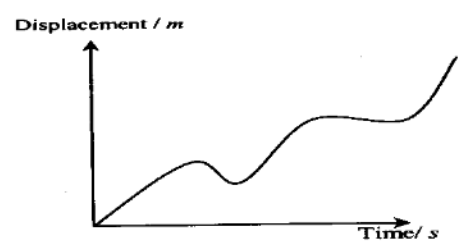
Acceleration: This is the rate of change of velocity
A body is said to be accelerating when its velocity increases, and is said to be decelerating when its velocity is decreasing. However, if the velocity of a body is constant, then its acceleration is zero. The SI unit of acceleration is ms^{-2}
Uniform acceleration: this happens if the velocity of a body changes with equal amounts in equal successive time intervals, no matter how small the time intervals may be.

Displacement time graphs

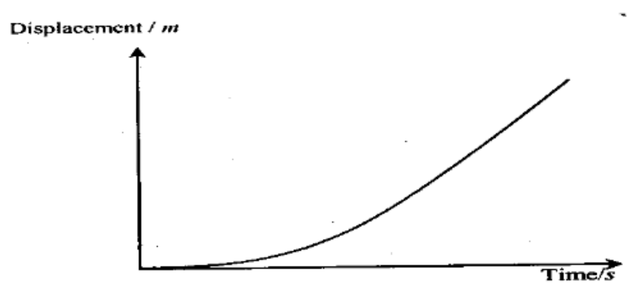
a) *Uniform velocity*



b) *Non uniform velocity*

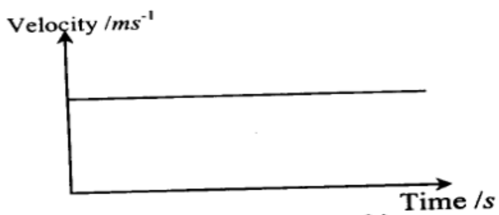


c) *Uniform acceleration*

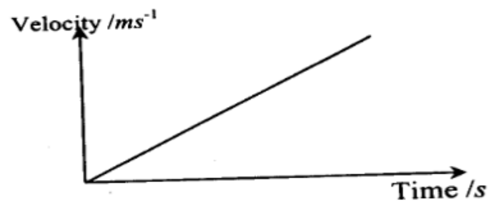


Velocity-time graphs

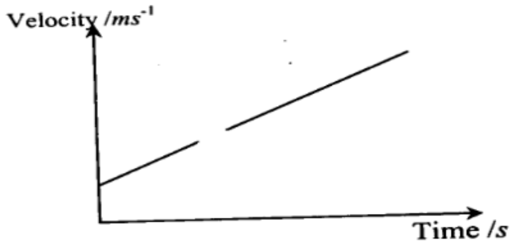
a) *Zero acceleration (uniform velocity)*



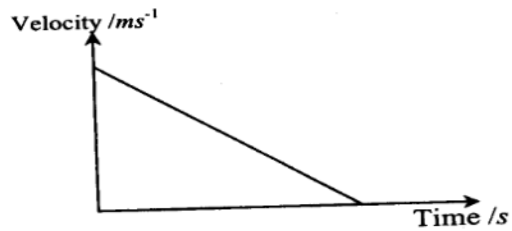
b) *Uniform acceleration (with $u = 0$)*



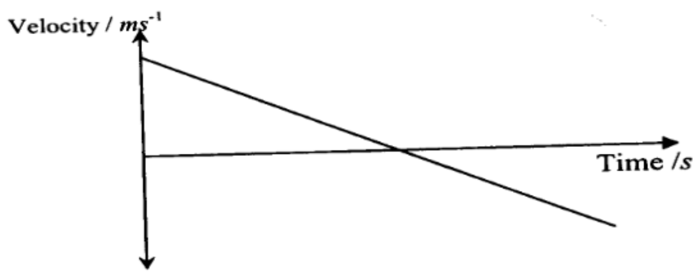
c) *Uniform acceleration ($u \neq 0$)*



d) *Uniform deceleration*



e) *Motion of a body thrown upwards, such that it then falls with uniform acceleration.*



Equations of motion

If the velocity of a body changes from u to v in a time t, then from the definition of acceleration, $a = \frac{v-u}{t}$

$$\Rightarrow at = v - u$$

$$\therefore v = u + at \qquad \textit{this is the first equation of motion}$$

As the velocity changes steadily,

Average velocity $= \frac{u+v}{2}$

From displacement = $(average\ velocity) \times (time)$

$s = \left(\frac{u+v}{2}\right) \times (t)$ but from the first equation, $v = u + at$

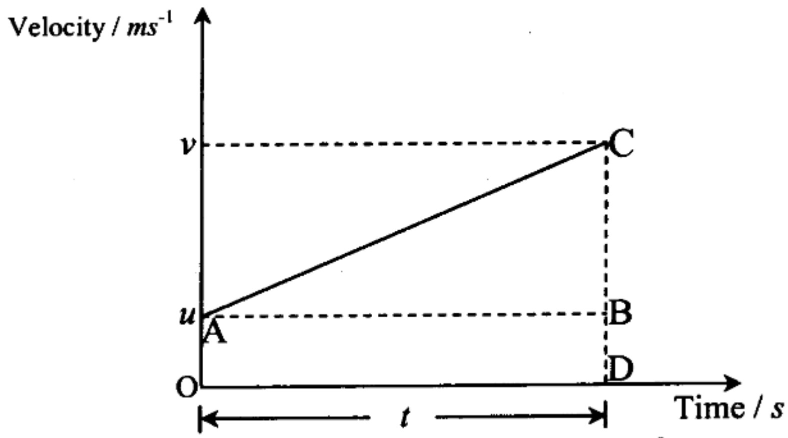
$$\Rightarrow s = \frac{[u+(u+at)]}{2} \times (t)$$

$$= \frac{2ut+at^2}{2}$$

$\therefore s = ut + \frac{1}{2}at^2$ this is the second equation of motion

Alternatively;

Consider a body with initial velocity u and final velocity v as it moves from A to B.



Displacement = total area under curve

= Area of OABD + Area of ACB

Area of OABD = ut

Area of ABC = $\frac{1}{2} \times t \times (v-u)$ $but\ \ v = u + at$

$\Rightarrow Area\ of\ ABC = \frac{1}{2}at^2$

$\therefore Total\ area = ut + \frac{1}{2}at^2 = displacement$

$\Rightarrow\ s = ut + \frac{1}{2}at^2\quad as\ before$

From $v = u + at,\ t = \frac{v-u}{a}$

But $s = \left(\frac{u+t}{2}\right) \times t \ \Rightarrow s = \left(\frac{u+v}{2}\right) \times \left(\frac{v-u}{a}\right)$

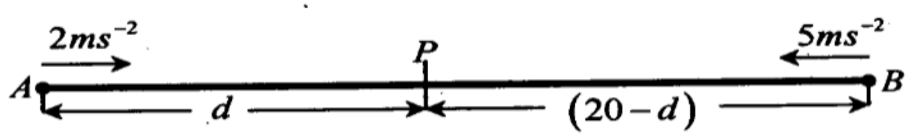
$s = \frac{uv - uv - u^2 + v^2}{2a} \Rightarrow s = \frac{v^2 - u^2}{2a}$

$\therefore v^2 = u^2 + 2as$, this is the third equation of motion

Examples

1. Two particles are travelling along a straight line AB of length 20m. At the instant when one particle tarts from rest at A and travels towards B with a constant acceleration of $2ms^{-2}$, the other starts from rest at B and travels towards A with a constant acceleration of $5ms^{-2}$. Find the time after which the two bodies collide and how far from A they collide.

Solution



Let the particles collide at a point P, a distance d from A

$s = ut + \frac{1}{2}at^2$

Particle from A

Particle from B

$u = 0$
$s = d$
$a = 2$
$\Rightarrow d = \frac{1}{2} \times 2 \times t^2 \ldots\ldots (i)$

$u = 0$
$s = 20 - d$
$a = 5$
$20 - d = \frac{1}{2} \times 5 \times t^2 \ldots\ldots. (ii)$

From (i) $d = t^2$

Substituting for d in (ii) gives:

$20 - t^2 = 2.5t^2$

$\Rightarrow 20 = 3.5t^2$

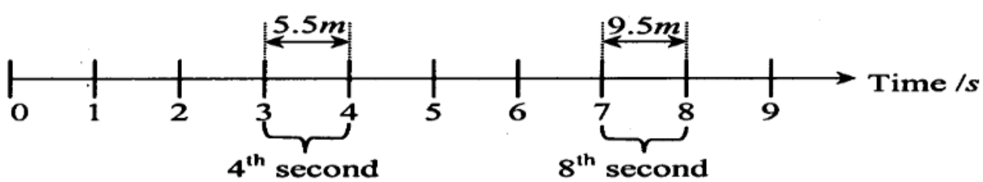
$\therefore t = 2.39s$

$d = 2.39^2 = 5.714m$

They collide 5.714m from A after 2.39s

2. A motorcar moving with a uniform acceleration covers 5.5m in its 4th second and 9.5m in its 8th second of its motion. Find its acceleration and initial velocity.

Solution



Distance covered in 4th second = (distance covered after 4s) – (distance covered after 3s)

$$\Rightarrow 5.5 = \left(4u + \frac{1}{2} \times a \times 4^2\right) - \left(3u + \frac{1}{2} \times a \times 3^2\right)$$

Multiplying through by 2 gives;

$11 = 8u + 16a - 6u - 9a$

$11 = 2u + 7a \ldots\ldots\ldots\ldots\ldots (i)$

Distance covered in 8th second = (distance covered after 8s) – (distance covered after 7s)

$$\Rightarrow 9.5 = \left(8u + \frac{1}{2} \times a \times 8^2\right) - \left(7u + \frac{1}{2} \times a \times 7^2\right)$$

Multiplying through by 2 gives;

$19 = 16u + 64a - 14u - 49a$

$19 = 2u + 15a \ldots\ldots\ldots\ldots\ldots (ii)$

Subtracting equations (ii) and (i) gives:

$8 = 8a \Rightarrow a = 1ms^{-2}$

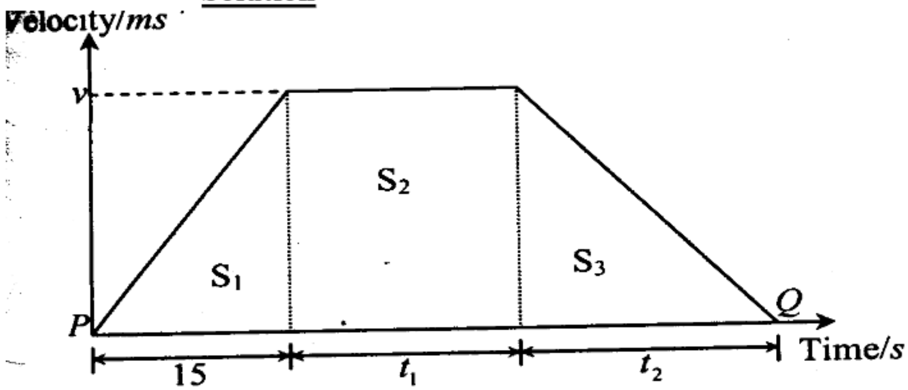
Substituting for a in equation (ii) gives;

$2u = 19 - 15(1)$

$2u = 4 \Rightarrow u = 2ms^{-1}$

3. A train stops at two stations P and Q, which are 2km apart. It accelerates uniformly from P at $1ms^{-2}$ for 15s, and maintains a constant speed for a time before decelerating uniformly to rest at Q. If the deceleration is $0.5ms^{-2}$, find the time for which the train is travelling at a constant speed.

Solution



$$s_1 + s_2 + s_3 = 2000 \ldots\ldots\ldots\ldots (i)$$

From $v = u + at$

$v = 0 + (1 \times 15) = 15\text{ms}^{-1}$

For the deceleration, From $v = u + at$, $0 = 15 + -0.5t_2$

$\Rightarrow t_2 = 30s$

$s_1 = \frac{1}{2} \times 15 \times 15 = 112.5m$

$s_2 = 15 \times t_1 = 15t_1$

$s_3 = \frac{1}{2} \times 30 \times 15 = 225m$

Substituting for $s_1, s_2\ and\ s_3$ in equation (i) gives;

$112.5 + 225 + 15t_1 = 2000$

$\therefore 15t_1 = 1662.5$

$\Rightarrow t_1 = 110.83s$

∴ The time for which the train is travelling at a constant speed = 110.83s

4. A car A, travelling at a constant velocity of 25ms^{-1}, overtakes a stationery car B. Two seconds later, car B sets off in pursuit, accelerating at a uniform acceleration 6ms^{-2}. How far does B travel before catching up with A?

Solution

A

$u = 25ms^{-1}, time = t, a = 0$

$s = ut + \frac{1}{2}at^2$

$\therefore s = 25t$ ………(i)

B

$u = 0ms^{-1}, time = t - 2, a = 6ms^{-2}$

$s = ut + \frac{1}{2}at^2$

$\therefore s = 3(t-2)^2$……..(ii)

Equations (i) and (ii) gives:

$25t = 3(t-2)^2$

$\Rightarrow 3t^2 - 37t + 12 = 0$

$\therefore (t-12)(3t-1) = 0 \Rightarrow either\ t = 12\ or\ t = \frac{1}{3}$

Note that $t = \frac{1}{3}$ cannot work, since the time taken by B will be $\left(\frac{1}{3} - 2\right)$ which gives a negative.

Therefore time taken by A before B catches it is 12 seconds.

From equation (i), $s = ut = 25 \times 12 = 300m$

Note that the same answer would be got by substituting for t in equation (ii).

Trial questions

1. A car is being driven along a road at a steady speed $25ms^{-1}$ when the driver suddenly notices that there is a fallen tree blocking the road 65m ahead. The driver immediately applies the brakes giving the car a constant retardation of $5ms^{-2}$. How far in front of the tree does the car come to rest?
   If the driver had not reached immediately and the brakes were applied one second later, with what speed would the car have hit the tree? [Ans: s = 2.5m, v = $15ms^{-1}$]

2. A, B and C are three points which lie in that order on a straight road with $\overline{AB} = 95m$ and $\overline{BC} = 80m$. A car is travelling along the road in the direction $\overrightarrow{ABC}$ with a constant acceleration a ms^{-2}. The car passes through A with a speed u ms^{-1}, reaches B five seconds later, and C two seconds after that. Calculate the values of u and a
   [Ans: u = 4 ms^{-1} , a = 6 ms^{-2}]
3. A train of mass 100,000kg starts from rest at station P and accelerates uniformly at $1ms^{-2}$ until it attains a speed of 30 ms^{-1}. It maintains this speed for further 90s and then brakes are applied, producing a resultant breaking force of 50KN. If the train comes to rest at station Q, Find the distance between the two stations. [Ans: 4050m]
4. A ship of mass 10^7kg is travelling at $2ms^{-1}$ when its engine is switched off. As a consequence, the ship's speed is reduced to $1.5ms^{-1}$ in a distance of 100m. Assuming that resistance to the ship's motion is uniform, calculate the magnitude of the resistance.
   [Ans: 87.5kN]
5. A train starts from rest at station A and accelerates at $1.25ms^{-2}$ until it reaches a speed of $20ms^{-1}$. It then travels at this steady speed for a distance of 1.56km and then decelerates at $2ms^{-2}$ to come to rest at B. Find the distance from A to B. [Ans: 1.82km]
6. Two cars A and B are traveling along a straight path. The cars are observed to be side by side when they are at point P of the path and again at another point Q. Assuming that A and B moved with a uniform acceleration a_1 and a_2, prove that if their velocities are U_1 and U_2 respectively, the distance PQ is given by; $\frac{2(U_1-U_2)(U_2a_1-U_1a_2)}{(a_1-a_2)^2}$

Motion under gravity

In absence of air resistance, all bodies regardless of their masses and composition fall with the same acceleration, if they are at the same location near the earth's surface. Such a fall is called free fall.

Acceleration due to gravity, g

This is the rate of change of velocity of a freely falling body

Note:

- Unless otherwise stated, the value of acceleration due t gravity is $9.81ms^{-2}$
- G replaces a in the equations of motion
- g is positive for a body falling and negative for a body moving upwards

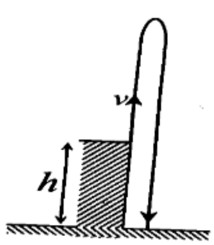
Note: For questions that have motion under gravity, we use arrow convention, such that ; before substituting numerical values in the formulae for uniformly accelerated motion, they should represent vectors in the same direction, or the arrows of the vectors should be in the same direction. The table below, as well as the following examples will make this point more clear.

	$\uparrow upward\ motion$	$\downarrow downward\ motion$
v	Negative	Positive
u	Positive	Negative
g	Negative	Positive
h	Negative	Positive

Example

1. A stone is thrown vertically upwards with a speed of $20ms^{-1}$ from a point at a height h above the ground level. If the stone hits the ground 5s later, find the;
   (i) velocity with which it hits the ground
   (ii) the value of h

Solution



Considering upward motion

From $v = u + at$

$-v = 20 + (-9.81 \times 5)$

$v = 29ms^{-1}$

From $s = ut + \frac{1}{2}at^2$

$-h = (20 \times 5) + \frac{1}{2} \times -9.81 \times 5^2$

h = 22.63m

2. A stone is thrown vertically upwards from the top of a tower and hits the ground 10s later, with a speed of 51ms^{-1}. Find the initial velocity and height of the tower.

Solution

$v = u + at$

Considering downward motion, $v = -u + at$

$\Rightarrow 51 = -u + 9.81 \times 10$

$\Rightarrow u = 47.1$ ms^{-1}

$s = ut + \frac{1}{2}at^2$

$\therefore s = (-47.1 \times 10) + \frac{1}{2} \times 9.81 \times 10^2$

$\Rightarrow s = 19.5m$

3. A ball is thrown vertically upwards with a speed of 15ms^{-1}, from a point which is 0.7m above the ground. Find the speed with which the ball hits the ground, and the time taken.

Solution

Considering the downward motion

$from\ v^2 = u^2 + 2as$

$v^2 = (-15)^2 + 2 \times 9.81 \times 0.7$

$v = 15.4$ms^{-1}

From $v = u + at$

$15.4 = -15 + 9.81t$

$t = 3.1s$

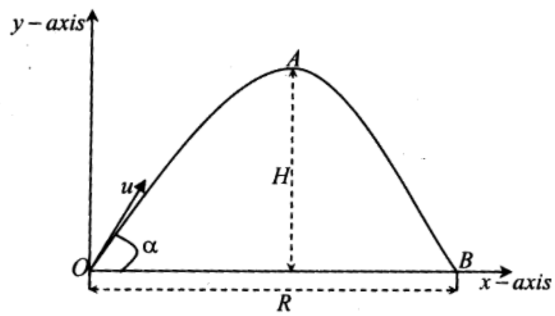
Trial questions

1. A stone is dropped vertically from the top of an overlapping cliff, and hits the sea 3 seconds later. The acceleration due to gravity at that location is 10ms^{-2}. Find the speed of the stone as it hits the sea, and the height of the cliff. [Ans: v = 30ms^{-1}, h = 45m]
2. A ball is projected from a point 2m above the ground with an upward speed of 3ms^{-1}. Assuming that the acceleration due to gravity is 10ms^{-2},find the;
   (i) time taken for the ball to reach its greatest height above the ground
   (ii) maximum height reached
   (iii) speed of the ball when it first strikes the ground
   [Ans: (i) 0.3m (ii) 0.45m (iii) 7ms^{-1}]
3. Two stones are thrown from the same point at the same time, one vertically upwards with a speed of 40 ms^{-1}, and the other vertically downwards at 40 ms^{-1}. Find how far apart the stones are after two seconds. [Ans: 160 m]
4. A ball is projected vertically upwards with a speed of 50ms^{-1}. On return it passes the point of projection and falls 78m below. Calculate the total time taken [Ans: 11.57s]

PROJECTILES

A projectile is anything which when given an initial velocity can be left to move on its own in presence of a constant force field, such as the gravitational force field. The effects of air resistances are neglected

Consider motion of a particle of mass, projected at an angle α to the horizontal with initial velocity u



The particle has got both horizontal and vertical motion

$Horizontal\ motion (x - axis)$ | $Vertical\ motion\ (y - axis)$

$Initial\ velocity, u_x = u\cos\alpha \ldots\ldots\ldots (i)$ | $initial\ velocity, u_y = u\sin\alpha \ldots\ldots (ii)$

$Acceleration, a_x = 0 \ldots\ldots\ldots (iii)$ | $Acceleration, a_y = -g \ldots\ldots\ldots\ldots (iv)$

From $v = u + at$

$v_x = u\cos\alpha + (0 \times t)$

$= u\cos\alpha \ldots\ldots (v)$ | $v_y = u\sin\alpha - gt \ldots\ldots (vi)$

From $s = ut + \frac{1}{2}at^2$

$x = ut\cos\alpha + \frac{1}{2} \times 0 \times t^2$

$x = ut\cos\alpha \ldots\ldots\ldots (vii)$ | $y = ut\sin\alpha - \frac{1}{2}gt^2 \ldots\ldots (viii)$

From equation (vii), $t = \frac{x}{u\cos\alpha}$

Substituting for t in equation (viii) gives;

$$y = u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$= \frac{xu\sin\alpha}{u\cos\alpha} - \frac{gx^2}{2u^2cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2u^2cos^2\alpha} \ldots\ldots\ldots (ix)$$

But $\frac{1}{cos^2\alpha} = sec^2\alpha \Rightarrow y = x\tan\alpha - \frac{gx^2sec^2\alpha}{2u^2} \ldots\ldots\ldots (x)$

$$= x\tan\alpha - \frac{gx^2(1+tan^2\alpha)}{2u^2} \ldots\ldots\ldots (xi)$$

Equations (ix), (x) and (xi) are equations of the parabola. They are called trajectory equations and hence the projectile describes a parabolic path.

Definitions:

Angle of projection, α: This is the angle between the direction of the projectile and the horizontal.

OAB is called the **trajectory** and it is the path taken by the projectile.

Greatest height reached, H: This is the distance between the highest point reached by the projectile and the horizontal plane through the point of projection.

From $v^2 = u^2 + 2as$

At maximum height, v = 0, $\therefore 0 = (u\sin\alpha)^2 - 2gH$

$$\Rightarrow 2gH = u^2 sin^2\alpha$$

$$H = \frac{u^2 sin^2\alpha}{2g}$$

To get the time taken to reach the maximum point, we use $v = u + at$

$$\therefore 0 = u\sin\alpha - gt$$

$$\Rightarrow t = \frac{u\sin\alpha}{g}$$

Time of flight, T: This the time taken by the particle(projectile) to move from its initail position to its final positions along its paths.

From equation (viii), $y = ut\sin\alpha - \frac{1}{2}gt^2$

At B, y = 0 (see diagram)

$$\Rightarrow 0 = uT\sin\alpha - \frac{1}{2}gT^2$$

$$\therefore 0 = T(u\sin\alpha - \frac{1}{2}gT)$$

Either T = 0 i.e at O ; or $u\sin\alpha - \frac{1}{2}gT = 0$

$$\Rightarrow u\sin\alpha = \frac{1}{2}gT$$

$$T = \frac{2u\sin\alpha}{g}$$

It should therefore be noted from the expression of time of flight, T and that of time taken to reach the maximum height, t that; T = 2t.This shows that the trajectory is symmetrical.

Horizontal range, R: This is the distance from the initial position of the projectile to its final position on the horizontal plane through the point of projection.

R = OB, and at B, $R = x,\ t = T = \frac{2u\sin\alpha}{g}$

From $x = ut\cos\alpha$,

At B, $R = uT\cos\alpha \quad \Rightarrow R = u\left(\frac{2u\sin\alpha}{g}\right)\cos\alpha$

$$\therefore R = \frac{2u^2\sin\alpha\cos\alpha}{g} \qquad but\ 2\sin\alpha\cos\alpha = \sin 2\alpha$$

$$\Rightarrow R = \left(\frac{u^2\sin 2\alpha}{g}\right)$$

For maximum horizontal range, R_{max}, we get the minimum value of $R = \left(\frac{u^2\sin 2\alpha}{g}\right)$

It can therefore be seen that R can have a maximum value if $\sin 2\alpha = 1$, because the maximum value of sine is 1

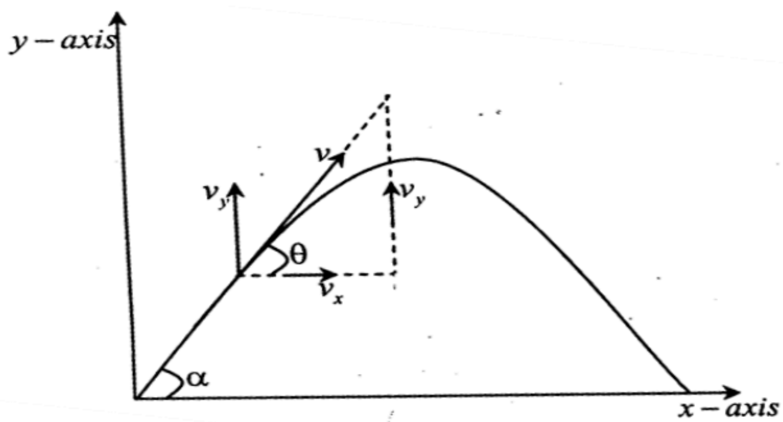
$$\therefore R_{\max} = \frac{u^2}{g}$$

This happens when $\sin 2\alpha = 1 \Rightarrow 2\alpha = 90^0$ *and so* $\alpha = 45^0$

$\therefore$ For maximum horizontal range, angle of projection should be 45^0

Direction of motion

This is determined by the velocity of the projectile at a particular time.



$$v^2 = {v_y}^2 + {v_x}^2$$

$$\Rightarrow v = \sqrt{{v_y}^2 + {v_x}^2} = magnitude\ of\ velocity\ at\ any\ time\ t$$

$$\tan\theta = \frac{v_y}{v_x} \quad \Rightarrow \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

But $v_y = u\sin\theta - \text{gt}$ *and* $u_x = u\cos\theta$

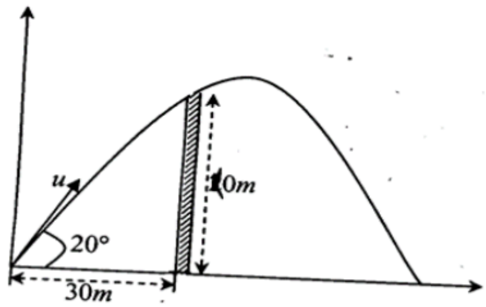
$$\therefore \theta = \tan^{-1}\left(\frac{u\sin\theta - \text{gt}}{u\cos\theta}\right)$$

θ is the angle which the direction of motion of the particle makes with the horizontal at that time, t. If $\tan\theta$ is positive, then θ is above the horizontal, and hence the particle has not yet reached the maximum point. If $\tan\theta$ is zero, then the particle is at the maximum point. If $\tan\theta$ is negative, then θ is below the horizontal, and hence the particle is falling after passing the maximum point.

Examples

1. A particle is projected at 20^0 to the horizontal and just clears a wall 10m high and 30m away from the point of projection. Find the speed of the projection, velocity of the projectile when it strikes the building and the time taken to reach the building.

Solution



Initial velocity:

From $y = x\tan\alpha - \frac{gx^2}{2u^2cos^2\alpha}$

$10 = 30 \times \tan 20 - \frac{9.81 \times 30^2}{2u^2cos^2 20}$

$\Rightarrow 10 - 10.92 = \frac{8829}{1.766u^2}$

$\therefore u = 73.76\text{ms}^{-1}$

Time taken:

From $x = ut\cos\alpha$,

$30 = 73.76 \times t \times \cos 20$

$\Rightarrow t = \frac{30}{73.76\cos 20} = 0.43s$

Velocity:

At the point when the particle strikes the building,

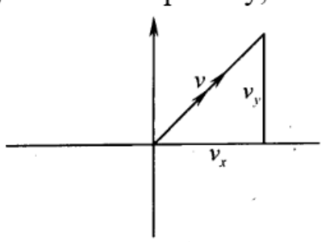
$v_x = u\cos\alpha = 73.76 \times \cos 20 = 69.3\text{ms}^{-1}$

$v_y = u\sin\alpha - gt = (73.76 \times \sin 20 - 9.81 \times 0.43) = 21\text{ms}^{-1}$

$v = \sqrt{{v_y}^2 + {v_x}^2} = \sqrt{69.3^2 + 21^2}$

$= 72.4\text{ms}^{-1}$

Since velocity is a vector quantity, we need to also find its direction



$\tan\theta = \frac{v_y}{v_x} = \frac{21}{69.3} \Rightarrow \theta = 16.86^0$

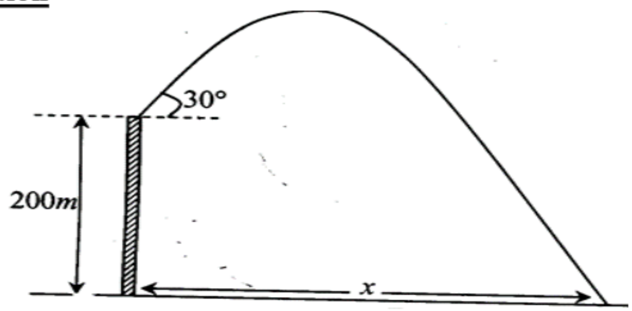
2. A shot is fired from the top of a cliff 200m high with a velocity of 500ms^{-1} at an elevation of 30^0. Find;

(i) the distance from the point where the shot strikes the ground to the bottom of the cliff

(ii) the time taken

(iii)The distance from the ground to the highest point reached

Solution



(i) From $y = ut\sin\alpha - \frac{1}{2}gt^2$

Considering upward motion:

$$-200 = 500t\sin 30 - \frac{1}{2}\times 9.81\times t^2$$

$$\Rightarrow 0 = 4.905t^2 - 250t - 200$$

$$t = \frac{250 \pm \sqrt{250^2 - (4\times 4.905\times(-200))}}{2\times 4.905}$$

Since t can not be a negative, $t = \frac{250+257.95}{9.81} = 51.76s$

(ii) From $x = ut\cos\alpha$,

$$x = 500\times 51.76\times\cos 30$$

= 22412.1m

(iii) From $H = \frac{u^2 sin^2\alpha}{2g}$

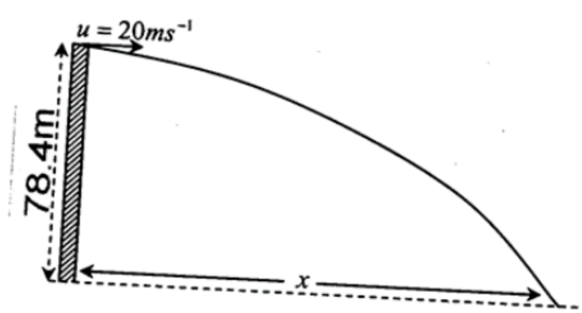
$$H = \frac{500^2 sin^2 30}{2\times 9.81} = 3185.5m$$

Required distance = 200 + 3185.5

= 3385.5m

3. A particle is projected horizontally at 20ms^{-1} from a point 78.4m above the horizontal surface. Find the time taken for the particle to reach the surface and the horizontal distance travelled in that time.

Solution



From $y = ut + \frac{1}{2}at^2$

Since the initial velocity is horizontal,u for vertical motion is zeroo

$$\Rightarrow 78.4 = \frac{1}{2}\times 9.81\times t^2$$

$$t \approx 4s$$

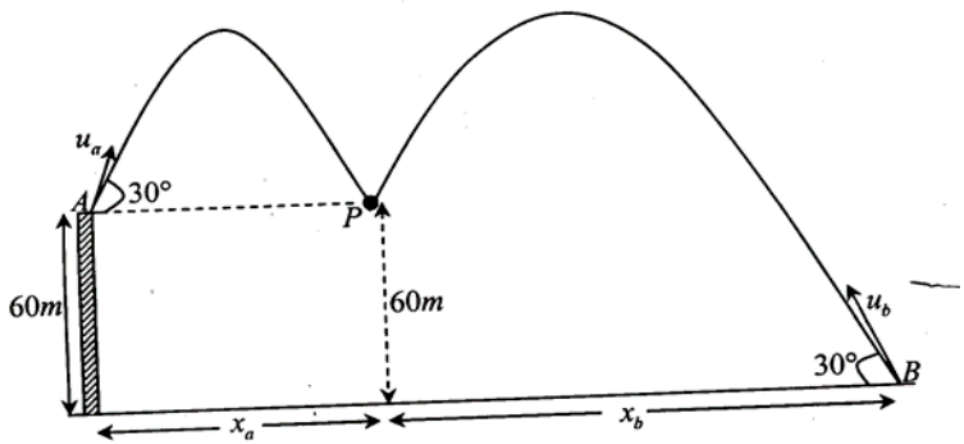
considering horizontal motion, a = g = 0

$$\Rightarrow x = ut$$

$$\therefore x = 20 \times 4 = 80m$$

4. An object A is projected upwards from a height 60m above the ground with a velocity of $20ms^{-1}$ at 30^0 to the horizontal. At the same time, another object B is projected from the ground upwards towards A at 30^0 to the horizontal. A and B collide at height of 60m above the horizontal ground, when they are both moving downwards, Find;
   (i) The speed of projection of B
   (ii) The horizontal distance between the points of projection
   (iii) The kinetic energy of A just after collision with B, if mass of A = 0.5kg

Solution



(i) If the particles collide at P, and considering A at the point of coincidence, the vertical displacement,

$$y = 0, \quad \alpha = 30 \quad and\ u = u_\alpha = 20$$

From $y = ut\sin\alpha - \frac{1}{2}gt^2$

$$\Rightarrow 0 = 20t\sin 30 - \frac{1}{2} \times 9.81 \times t^2$$

Either t = 0 , or $\frac{1}{2} \times 20 - \frac{1}{2} \times 9.81 \times t = 0$

$$\Rightarrow t = 2.04s$$

Since the particles were projected at the same time, they should take the same time to collide.

Considering particle B at the point of coincidence,

The vertical displacement, $y = 60, \alpha = 30 \quad and\ t = 2.04$

From $y = ut\sin\alpha - \frac{1}{2}gt^2$

$$\Rightarrow 60 = u_b \times 2.04 \times \sin 30 - \frac{1}{2} \times 9.81 \times (2.04)^2$$

$$60 = 1.02u_b - 20.4$$

$$\therefore u_b = 78.8ms^{-1}$$

(i) From $x = ut\cos\alpha$

$$x_a = 20 \times 2.04 \times \cos 30 = 35.33m \quad and$$

$x_b = 78.8 \times 2.04 \times \cos 30 = 139.25m$

Total distance between them = $x_a + x_b = 35.33 + 139.25 = 174.58m$

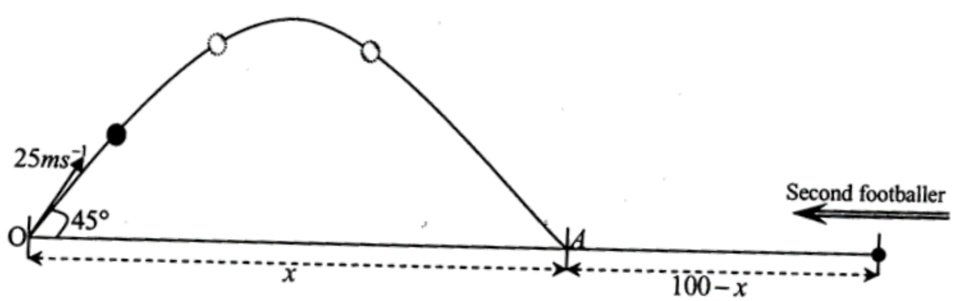
(ii) $Kinetic\ energy = \frac{1}{2}mv^2$

But velocity of A after collision at P = 20ms-1, but in opposite direction

$$\Rightarrow K.e = \frac{1}{2} \times 0.5 \times 20^2 = 100Joules$$

5. Two footballers 100m apart, stand facing each other. One of them kicks the ball from the ground such that the ball takes off at a velocity of $25ms^{-1}$ at 45^0 to the horizontal. Find the speed at which the second footballer should run towards the first baller in order to trap the ball as it touches the ground, if he starts running at the instant the ball is kicked.

Solution



Consider the ball: At the point A, the vertical displacement, y = 0

∴From $y = ut\sin\alpha - \frac{1}{2}gt^2$

$\Rightarrow 0 = 25t\sin 45 - \frac{1}{2} \times 9.81 \times t^2$

Either t = 0 i.e at O or $25\sin 45 - \frac{1}{2} \times 9.81 \times t = 0$

$\therefore 4.9t = 25 \times \sin 45 \quad \Rightarrow \quad t = 3.6s$

From $x = ut\cos\alpha$

$x = 23 \times 3.6 \times \cos 45 = 63.63m$

Consider the second footballer:

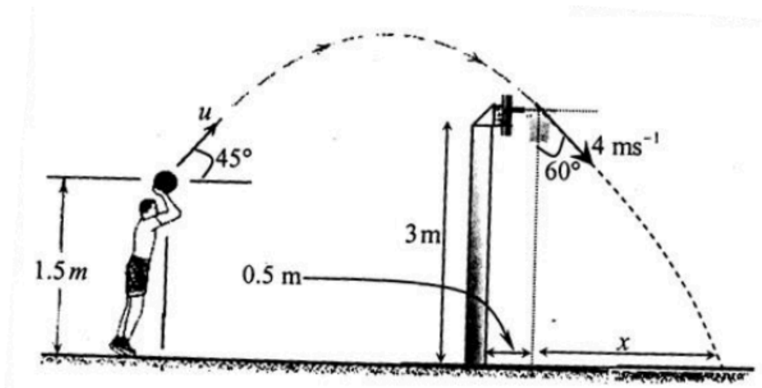
He is supposed to travel a distance of $(100 - x) = 100 - 63.63 = 36.37m$

Since the second footballer starts running at the instant the ball is kicked and is supposed to run so as to trap the ball as it falls, he should take the same time as that taken by the ball to land.

⇒ He should take 3.6s

$\therefore From\ speed = \frac{Distance}{time} = \frac{36.37}{3.6} = 10.1ms^{-1}$

6. In the following figure, a ball is projected with a speed u at an angle of 45^0 to the horizontal from a point 1.5m above the ground as shown above. It passses through a horizontal ringwith a velocity of $4ms^{-1}$, at an angle of 60^0 to the axis of the ring. If the axis of the ring is 0.5m and that the height of the ring is 3m,



calculate:

(i) Time taken by the ball to hit the ground from the instant it passes through the ring,

(ii) Distance between the pole and the point where the ball hits the ground

(iii) Initial speed of projection

Solution

(i) Starting from the ring: Let the velocity of the ball at the ring be v = $4ms^{-1}$

From $y = ut\sin\alpha - \frac{1}{2}gt^2$

$$3 = 4t\sin 30 - \frac{1}{2} \times 9.81 \times t^2$$

$$\Rightarrow 0 = 4.905t^2 + 2t - 3$$

Using $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and since t cannot be negative, t = 0.6s

(ii)

From $x = ut\cos\alpha$

$$x = 4 \times 0.6 \times \cos 30 = 2.08m$$

Required distance = 0.5 + x

$$= 0.5 + 2.08 = 2.58m$$

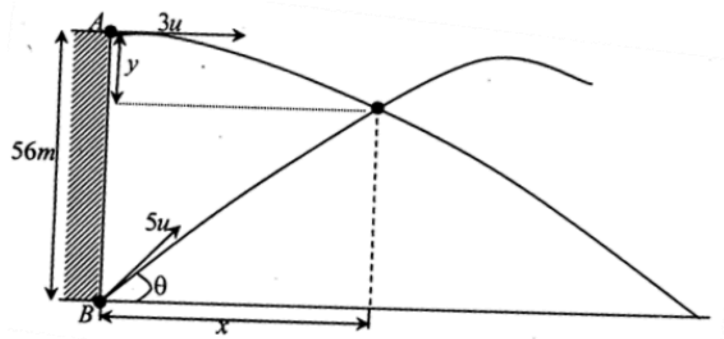
(iii) Since the horizontal velocity of the ball is the same through out its motion

Then $u\cos 45 = v\cos 30$

$$\therefore u = \frac{v\cos 30}{\cos 45} = \frac{4\cos 30}{\cos 45} \Rightarrow u = 4.89ms^{-1}$$

7. Two particles A and B are projected simultaneously, A from the top of a vertical cliff, and B from the base. Particle A is projected with a speed of 3u, while B is projected at an angle θ above the horizontal with a speed of 5u. The two particles collide after 2s. If the cliff is 56m high, find the;
   (i) Values of u and θ
   (ii) The horizontal and vertical distances from the base of the cliff to the point of collision of the two particles.

Solution



From $x = ut\cos\theta$

Considering particle A:

$$x = (3u) \times 2 \times \cos 0$$

$$\Rightarrow x = 6u \ldots\ldots\ldots (i)$$

Cosidering particle B:

$$x = (5u) \times 2 \times \cos\theta$$

$$\Rightarrow x = 10u\cos\theta \ldots\ldots\ldots (ii)$$

Equating the two equations,

$$6u = 10u\cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\frac{6}{10} = 53.13^0$$

From $y = ut\sin\alpha - \frac{1}{2}gt^2$

For particle A: (↓) $y = (3u) \times 2 \times \sin 0 - \frac{1}{2} \times 9.81 \times 2^2$

$$\Rightarrow y = 19.6m$$

∴ Vertical distance from the base to the point of coincidence

$$= 56 - 19.6 = 36.4m$$

For particle B,

$$36.4 = (5u) \times 2 \times \sin 53.13 - \frac{1}{2} \times 9.81 \times 2^2$$

$$36.4 = 8u - 19.6$$

$$u = 7\text{ms}^{-1}$$

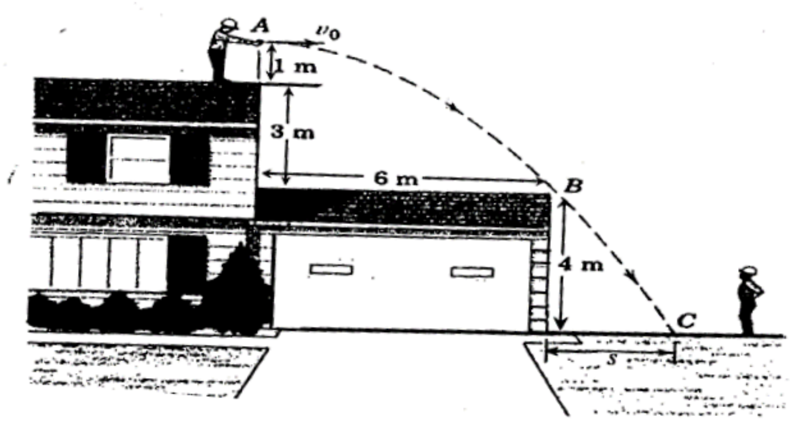
Substituting for u in equation (i) gives; $x = 6 \times 7 = 42m$

This is the horizontal diplacement of the particles at the point of collision

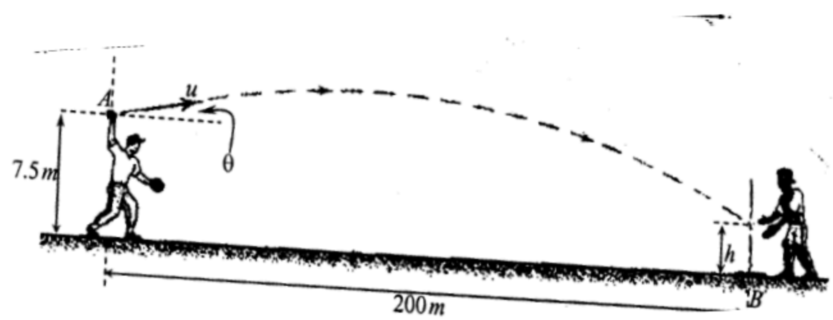
Trial questions

1. A dart player stands 3.0m from a small soft board of mass 0.20kg which is suspended freely. The player throws a dart of mass 0.05kg such that the dart leaves his hand with a horizontal velocity at a point 1.80m above the ground. The dart strikes the board at a point 1.50m from the ground. Assuming that air resistance is negligible, calculate the:
   (i) Time of flight of the dart,
   (ii) Initial speed of the dart
   (iii) Height to which the bottom of the soft board rises after the dart has embedded itself into it

2. The position vectorof a particle moving in the x-y plane at time t = 3.60s is 2.76i – 3.28j m. At t = 3.62s, its position vector is 2.79i – 3.33j m. Determine the magnitude v of its average velocity during this interval and the angle θ made by the average velocity with the x-axis. [Ans: $v = 2.92ms^{-1}$, $\theta = -59.0^0$]
3.



In the figure above, a roofer tosses a small tool towards a coworker on the ground. What is the minimum horizontal velocity v_0 necessary so that the tool clears point B? Calculate the distance s of the point of impact for the tool. [Ans: $v_0 = 6.64ms^{-1}$, s =2.49m]
4.



In the figure above, an outfielder experiments with two different trajectories for throwing to the home plate from the position shown:
(i) $u = 80ms^{-1}$ and $\theta = 10^0$ and
(ii) $u = 100ms^{-1}$ and $\theta = 15^0$
For each set of the initial conditions above, determine the time t required for the baseball to reach the home plate and the altitude h as the ball crosses the plate.
5. Calculate the range of a projectile which is fired at an angle of 45^0 to the horizontal with a speed of $20ms^{-1}$. [Ans: 40.77m]
6. A projectile is fired horizontally from the top of the cliff 250m high. The projectile lands 1.414×10^3m from the bottom of the cliff. Find the
(i) Initial speed of the projectile
(ii) Velocity of the projectile just before it hits the ground
[Ans: (i) $198ms^{-1}$ (ii) $210ms^{-1}$ at 19.5^0]

VECTORS AND SCALARS

- **Vector quantity :**

A vector quantity is that quantity which has both magnitude and direction
Examples include; velocity, momentum, Force, displacement, acceleration, weight, impulse, etc

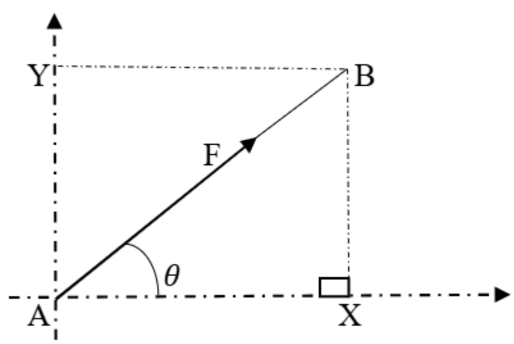
- **Scalar quantity:**

It is that quantity which has magnitude only
Examples include; Temperature, volume, mass, distance, speed, energy, power, charge , density etc

RESULTANT AND COMPONENTS OF FORCES

Resolving of forces

The component of the force F in any given direction is the effect of the force in that particular direction.
Consider a force F acting at an angle θ to the x-axis as shown below. Let AB represent the force F and angle BXA = 90^0, AX and AY represent the horizontal and vertical components of F.



$\frac{AX}{AB} = \cos\theta$ and $\frac{AY}{AB} = \sin\theta$

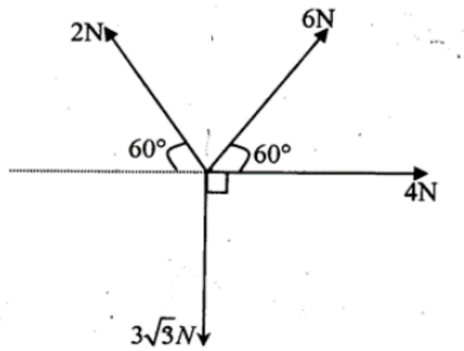
$AX = AB\cos\theta$ $AY = AB\sin\theta$

$AX = F\cos\theta$ $AY = F\sin\theta$

Hence the horizontal and vertical components are $F\cos\theta$ and $F\sin\theta$ respectively

Examples

1. Find the resultant of the forces shown in the figure below



Solution

Resolving,

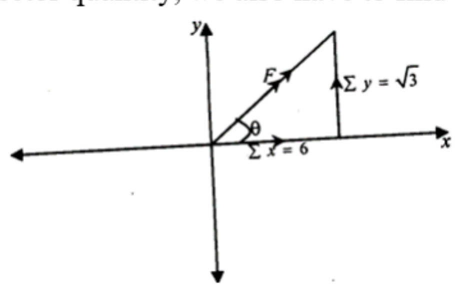
$Force, F$	$Horizontally (\rightarrow)$	$vertically\ (\uparrow)$
4	4 cos 0 = 4	4 sin 0 = 0
6	6 cos 60 = 3	6 sin 60 = $3\sqrt{3}$
2	2 cos 60 = 1	2 sin 60 = $\sqrt{3}$
3	$3\sqrt{3}$ cos 90 = 0	-$3\sqrt{3}$ sin 90 =-$3\sqrt{3}$
	$\sum_x = 6$	$\sum_y = \sqrt{3}$

$\sum x$ *and* $\sum y$ are the summations of the horizontal and vertical components of the resultant force.

If F is the resultant force, then,

$$F = \sqrt{x^2 + y^2} = \sqrt{6^2 + (\sqrt{3})^2} = 6.24\text{N}$$

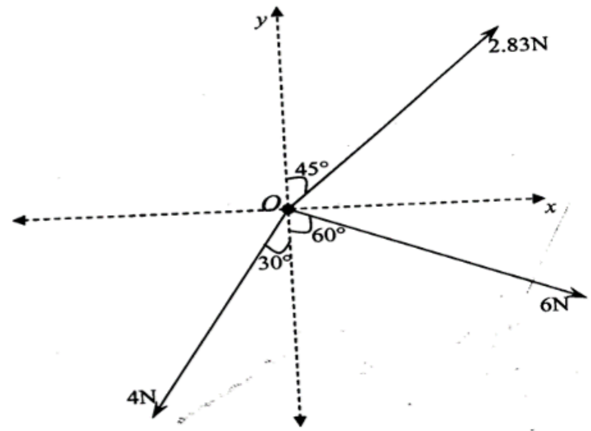
Since force is a vector quantity, we also have to find its direction



$$\tan\theta = \frac{\Sigma x}{\Sigma y} = \frac{\sqrt{3}}{6} \Rightarrow \theta = 16.1^0$$

Therefore, the resultant force is 6.24N and makes an angle of 16.1^0 to the horizontal

2. In the figure forces of 4N, 6N and 2.83N act on a particle O.
   (i) Find the resultant force,
   (ii) Find the acceleration of the particle if it has a mass 2kg

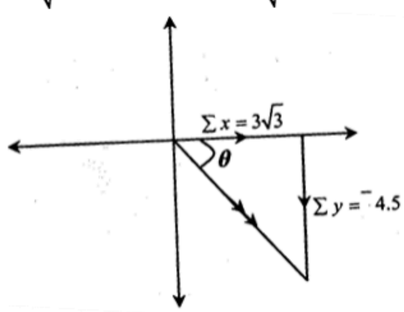


Solution

(i) Resolving,

$Force, F$	$Horizontally (\rightarrow)$	$vertically\ (\uparrow)$
4	- 4 cos 60 = -2	- 4 sin 60 = - 3.5
2.83	2.83 cos 45 = 2	2.83 sin 45 = 2
6	6 cos 30 = $3\sqrt{3}$	3 sin 30 = -3
	$\sum_x = 3\sqrt{3}$	$\sum_y = -4.5$

Resultant = $\sqrt{\sum x^2 + \sum y^2} = \sqrt{(3\sqrt{3})^2 + (-4.5)^2}$ = 6.88 N



$\tan\theta = \frac{\sum x}{\sum y} = \frac{4.5}{3\sqrt{3}} \Rightarrow \theta = 40.9^0$

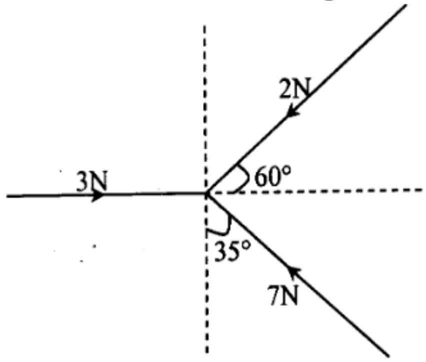
Therefore the resultant force is 6.88N and is 40.9^0 below the horizontal

(ii) Force $= mass \times acceleration$

$\Rightarrow 6.88 = 2a$

$\therefore a = 3.44$ms^{-2}

3. Find the resultant of the forces in the figure below



Solution

Resolving,

$Force, F$	$Horizontally (\rightarrow)$	$vertically\ (\uparrow)$
2	- 2 cos 60 = -1	- 2 sin 60 = - $\sqrt{3}$
3	3 cos 0 = 3	3 sin 0 = 0
7	-7 cos 55 = -4.02	7 sin 55 = -3
	$\sum_x =$ -2.02	$\sum_y = 4$

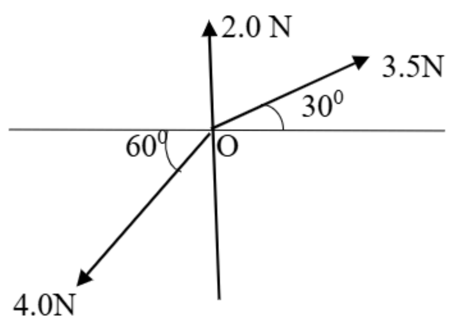
Resultant = $\sqrt{\sum x^2 + \sum y^2} = \sqrt{(-2.02)^2 + (4)^2}$ = 4.481 N

$$\tan\theta = \frac{\sum x}{\sum y} = \frac{4}{-2.02} \Rightarrow \theta = -63.21^0$$

The resultant is 4.48N at 63.21^0 below the horizontal

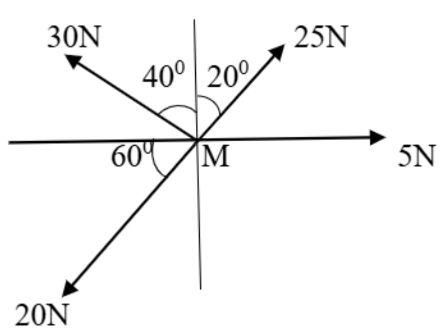
Trial questions

1.



Three forces of 3.5N, 4.0N and 2.0N act at a point O shown in the figure. Find the resultant force [Ans: 1.07N at 15.5^0]

2.



A body , M of mass 6kg is acted on by the forces of 5N, 20N, 25N and 30N as shown. Find the acceleration of M [Ans: 5.52 ms^{-2}]

Relative motion

This is the observed motion of one moving object with respect to another.
For a person in a moving vehicle, trees and buildings near the road appear to be moving in the opposite direction, yet those that are far appear to be moving in the same direction.
If a car B moving with velocity VB overtakes another car A moving with a velocity VA, the passenger in car A sees car B apparently moving towards him.However, the person in car B sees a gradual catching up. In this case, car A appears to be stationary as B overtakes it .
The velocity car B appears to have to an observer in car A is called the velocity of B relative to A.
Therefore the velocity of B relative to A is the resultant velocity of B when A appears stationary. One way of making a moving body stationery is by reversing its velocity. The velocity of B relative to A is written as;

$_BV_A$ or $V_{BA} = V_B - V_A$

Similarly

$_AV_B = V_A - V_B$ is the velocity of A relative to B

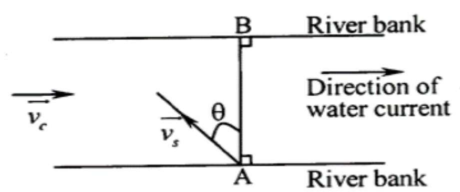
$\Rightarrow {_AV_B} = -{_BV_A}$

The direction in which a vessel or boat or air craft is heading is called the course. If the vessel is affected by wind or current (not in the same direction). It will be pushed off its course. The direction in which it actually moves is called track.
The angle between the course and the track is called drift.
The vessel's speed along the track is its speed relative to the ground or sea bed. This is called the vessel's resultant speed/ ground speed.

Resultant velocity

Since velocity is a vector quantity with both magnitude and direction, two velocities can be combined to form a single velocity. Such velocities are usually met in problems of crossing a river by a boat or swimmer.
Consider a problem of crossing a river from a point on one bank to a point on the other bank. Assuming that the banks are parallel, we shall consider two cases;

<u>**Case 1:**</u>
If one is to cross from point A on one bank to a point B directly opposite to A on the other bank, course set by the boat or swimmer must be upstream.

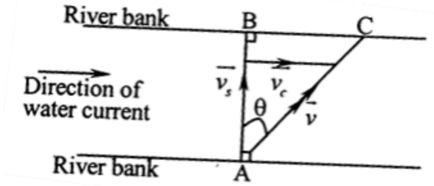


In the figure, $\vec{v_c}$ is the velocity of water current and $\vec{v_s}$ the velocity of the boat or swimmer in still water.

Resolving velocities; $v_s \sin\theta = v_c$

The resultant velocity in crossing the river is along AB and is given; $v_s \cos\theta$
It is instructive to note that in this case the boat or swimmer crosses the river by the shortest distance.

Case 2:
If the river is to be crossed by the shortest time possible (as quickly as possible), the course of the boat or swimmer is directly across (perpendicular) to the river bank, such that the water current carries the boat or swimmer down stream.



Resultant velocity,

$$v = \sqrt{{v_s}^2 + {v_c}^2}$$

This resultant velocity is at an angle θ to the horizontal, where $\tan\theta = \frac{v_c}{v_s}$

Time taken to cross the river $= \frac{\overline{AB}}{v_s} = \frac{\overline{AC}}{v}$

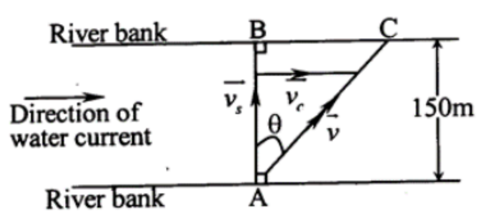
The distance $\overline{BC}$, which the boat or swimmer moves down stream $= v_c \times t$

Examples

1. A boy who can swim at $2ms^{-1}$ in still water wishes to swim across a river, 150m wide. If the river flows at $1.5ms^{-1}$, find;
   (i) The time the boy takes for the crossing and how far downstream he travels, if he is to cross the river as quickly as possible,
   (ii) The course that he must set in order to cross to a point exactly opposite the starting point, and the time taken for the crossing

Solution

(i)



$v_c = 1.5ms^{-1}$, $v_s = 2ms^{-1}$

Time taken $= \frac{\overline{AB}}{v_s} = \frac{150}{2} = 75s$

Distance down stream $= v_c \times t = 1.5 \times 75 = 112.5m$

(ii)

Let the course be set at an angle θ

$$v_s \sin\theta = v_c \Rightarrow \sin\theta = \frac{1.5}{3}$$
$$\therefore \theta = 48.6^0$$

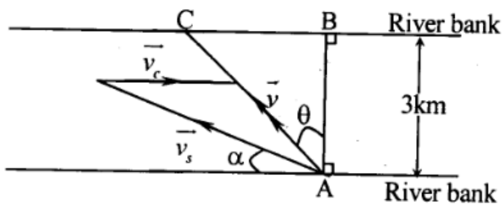
The course set is therefore upstream at an angle of $(90 - 48.6) = 41.4^0$ to the river bank

Speed across the river $= v_s \cos\theta = 2\cos 48.6$

Time taken $= \frac{150}{2\cos 48.6} = 113.41s$

2. A boat crosses a river 3km wide flowing at 4ms^{-1} to reach a point on the opposite bank 5km upstream. The boat's speed in still water is 12ms^{-1}. Find the direction in which the boat must be headed.

Solution



$$v = v_c + v_s \text{ and } \overline{BC} = 5km$$

In order to cross the river and reach a point C 5km upstream on the other bank, the course must be such that the resultant velocity v is in the direction AC.

From the diagram, $\tan\theta = \frac{5}{3} \Rightarrow \theta = 59^0$

Resolving;

$(\rightarrow)\quad -v\sin\theta = v_c - v_s\cos\alpha$

$\Rightarrow v\sin 59 = 12\cos\alpha - 4 \ldots\ldots..(i)$

$(\uparrow)\quad v\cos\theta = v_s\sin\alpha$

$\Rightarrow v\cos 59 = 12\sin\alpha \ldots\ldots\ldots\ldots\ldots(ii)$

Dividing equations (i) and (ii) gives:

$$\tan 59 = \frac{12\cos\alpha - 4}{12\sin\alpha} \Rightarrow 20\sin\alpha = 12\cos\alpha - 4$$

Dividing both sides by 4 and then squaring both sides gives;

$$25sin^2\alpha = 9cos^2\alpha - 6\cos\alpha + 1$$

But $sin^2\alpha = 1 - cos^2\alpha \Rightarrow 25(1 - cos^2\alpha) = 9cos^2\alpha - 6\cos\alpha + 1$

$$34cos^2\alpha - 6\cos\alpha - 24 = 0$$

This is a quadratic equation in $\cos\alpha$ which yields $\cos\alpha = 0.933$

$$\therefore \alpha = 21^0$$

⇒ the boat should be headed at 21^0 to the river bank upstream

Trial question

1. The banks of a river are parallel and 50m apart and a current flowsat $8ms^{-1}$. A bot with a speed of $10ms^{-1}$ in still water sails from a point A at one bank to point B directly opposite on the other bank.
   (i) Find the direction in which the boat is steered, and the time it takes to cross the river
   (ii) Explain why the shortest time of crossing is achieved when the boat is steered in a direction perpendicular to the banks, and find where the boat reaches on the opposite bank.

[Ans: $\alpha = 36.9^0, 8.33s,\ 40m$]

CHAPTER 3: NEWTON'S LAWS OF MOTION

First law

It states that; Everybody continues in its state of rest or of uniform motion in a straight line unless an external force is applied on it. This law expresses the idea of inertia

Inertia of a body is the reluctance of that body to start moving (if at rest) or to stop moving (after it has started moving)

Second law

It states that; The rate of change of momentum is directly proportional to the applied force and it takes place in the direction of force

If initial momentum = mu and final momentum = mv, then

Change in momentum = mv – mu

The rate of change of momentum = $\frac{mv-mu}{t}$

From the second law; $F \propto \frac{mv-mu}{t}$

$\Rightarrow F = k\frac{mv-mu}{t}$ where k is the constant of proportionality

$F = k\,m\left(\frac{v-u}{t}\right)$ but $\frac{v-u}{t} = acceleration, a$

$\Rightarrow F = k\,ma$

For a force of 1N, the acceleration caused on a body of mass 1kg is $1ms^{-2}$

$\therefore 1 = k \times 1 \times 1 \Rightarrow k = 1$

This gives F = ma

Third law

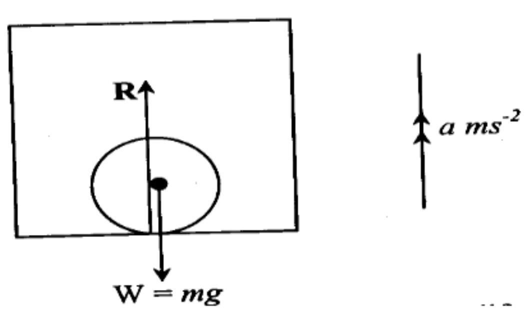
It states that: Action and reaction are equal and opposite. This means that if two bodies say A and B are placed together and A exerts a force (action) on B, then B exerts an equal and opposite force (reaction) on A.

Note that action and reaction forces occur in pairs and act on different bodies.

BODY IN A LIFT

Consider a body of mass m kg standing on the floor of a lift which is moving with an acceleration of a ms^{-2}

Case 1: when the lift is moving upwards

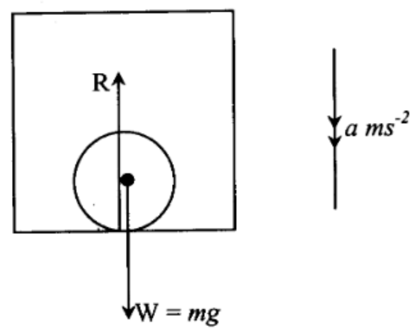


The resultant force that acts on the lift is given by; F = ma

Total upward force between the two forces gives the resultant force on the body in the lift, and since the lift is moving upwards, this resultant force should be acting upwards, implying that $R > mg$

$R - mg = ma$, $R = m(a + g)$

Case 2: when the lift is moving downwards



For downward motion, mg should be greater than R(mg > R), and still the difference between the two gives the resultant force.

$$mg - R = ma,$$
$$R = m(g - a)$$

Case 3: when the lift is moving either up or down with a constant velocity, then its acceleration is zero.

$$R - mg = m \times 0$$
$$R = mg$$

Examples

1. A man whose mass is 70kg stands on a spring weighing machine inside a lift. When the lift starts to ascend, its acceleration is $2.5ms^{-1}$
   (i) The reading of the weighing machine,
   (ii) The reading of the weighing machine when,
   (a) The velocity of the lift is uniform,
   (b) The lift comes to rest with a retardation of $5ms^{-1}$

Solution

(i) $R = m(g + a)$

$R = 70(9.81 + 2.8)$

$= 861.7N$

Note that the value of the normal reaction gives the reading of the weighing machine.

(ii) (a) When the velocity is uniform, acceleration = 0

$R = mg$

$R = 70 \times 9.81$

$= 686.7N$

(b) $a = -5ms^{-1}$

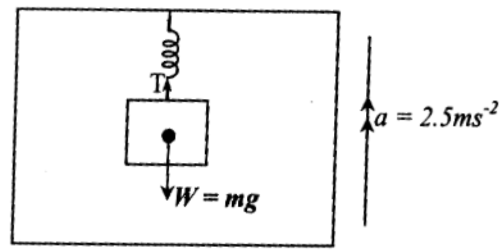
From $R = m(g + a)$,

$R = 70(9.81 - 5)$

$= 336.7N$

2. A body hangs from a screen balance supported from the roof of a lift. The lift has an upward acceleration of $2.5ms^{-2}$ and the balance reads 60N.

(i) What is the weight of the body?
(ii) Under what circumstances the balance will read 40N?
(iii) What would the balance read if the cable of the lift breaks?



Note that the reading of the machine gives the value of tension, T.

$T = 60N$

From $T - mg = ma$, $T = m(a + g)$

$60 = m(9.81 + 2.5)$ $\Rightarrow m = 4.87kg$

But, Weight, $W = mg$

$\Rightarrow W = 4.87 \times 9.81 = 4.87N$

(ii) From $T = m(g + a)$,

$40 = m(9.81 + a)$, but, $m = 4.87kg$

$40 = 4.87(9.81 + a)$

$40 = 47.78 + 4.87a$

$a = -1.596ms^{-2}$

Therefore, the balance will read 40N, when the lift is retarding at $1.596ms^{-2}$

(iii) The reading of the balance would be zero, and the lift would fall freely with acceleration due to gravity.

Trial questions

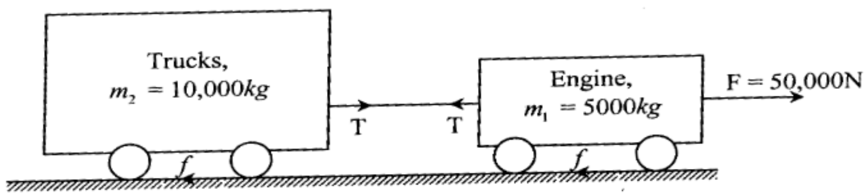
A lift of 950 kg is carrying a woman of mass 50kg

(a) The lift is ascending at a uniform speed. Calculate the :
  (i) Tension in the lift cable
  (ii) Vertical force exerted on the woman by the floor of the lift
(b) Sometime later, the lift is ascending with a downward acceleration of $2ms^{-2}$.Calculate the :
  (i) tension in the lift cable
  (ii) Vertical force exerted on the woman by the floor of the lift
  [Ans: (a) 9800N, 490N upwards (b) 7800N, 390N upwards]

Examples on application of F = ma

1. An engine of mass 5000kg pulls a train of 5 trucks each of mass 2000kg along a horizontal track. If the engine exerts a frictional force of 50,000N, and given that the frictional resistance is 5000N, calculate the
   (i) Acceleration
   (ii) Tension in the rope used for pulling

   Solution



For the engine

Resultant force = F – (T + f)

m_1a = F – (T + f)

5000 × a =50000 – T – 5000

5000a = 45000 – T …….(i)

for the trucks

Resultant force = (T – f)

m_2a = (T – f)

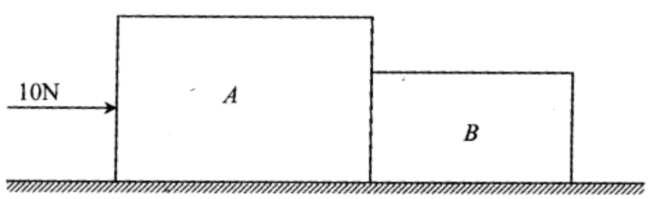
10000 × a = T – 5000

10000a = -5000 + T ……….(ii)

Solving the two equations simultaneously gives;

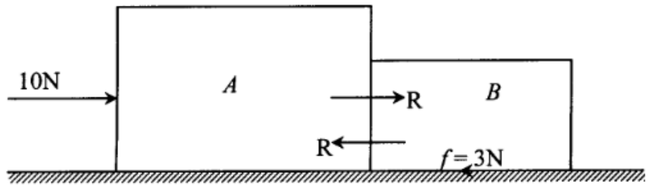
a = 2.67ms^{-2} and T = 31700N

2. In the figure below, two blocks A and B of masses 8kg and 4kg respectively are in contact on a horizontal table. A constant horizontal force of 10N is applied on block A as shown below. There is a constant frictional resistance of 3N between the table and block B but there is no frictional force between A and the table.



Calculate the: (i) acceleration of the blocks
(ii) Constant force between the two blocks

Solution



Considering body A

Resultant force = *10 – R*

5a = 10 – R …….(i)

considering body B

Resultant force = R – f

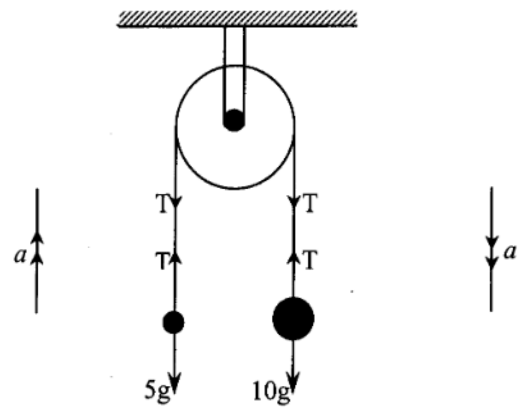
4a = R – 3 ……….(ii)

Solving the equations simultaneously gives

a = 0.778ms^{-2} R = 6.11N

3. Two particles of masses 5kg and 10kg are connected by an inextensible string which passes over a smooth fixed pulley. Find the acceleration of the system and the tension in the string.

   Solution



Considering the 5kg particle, R = T – mg $\Rightarrow 5a = T - 5g \ldots\ldots(i)$

Considering the 10kg particle, R = mg – T $\Rightarrow 10a = 10g - T \ldots\ldots(ii)$

Solving the two equations simultaneously, gives;

$a = \frac{1}{3}g = \frac{9.81}{3} = 3.27ms^{-2}$ and T = 65.4N

Trial questions

1. The motion of a lift, when ascending from rest, is in three stages. First, it accelerates at 1 ms^{-2} until it reaches a certain velocity. It then maintains this velocity for a period of time, after which it slows, with a retardation of 1.2ms^{-2}, until it comes to rest. Find the reaction between the floor of the lift and a passenger, of mass 100kg, during each of these three stages. [Ans: 1080N, 980N, 860N]
2. A lift travels vertically upwards from rest at floor A to rest at floor B which is 20m above A, in three stages as follows. At first, the lift accelerates from rest at A at 2 ms^{-2} for 2s. It then travels at a constant speed and finally it decelerates uniformly, coming to rest at B after a total time of 6.5s. Find the magnitude of the constant deceleration.

If the mass of the lift and all its contents is 500kg, find the tension in the lift cable during the stage of its motion when it is accelerating upwards [Ans:4ms^{-2} , 5900N]

3. Bodies of mass 6kg and 2kg are connected by a light inextensible string which passes over a smooth fixed pulley. With the masses hanging vertically, the system is released is released from rest. Find the acceleration of the system and the distance moved by the 6kg mass in the first two seconds of its motion [Ans:4.9ms^{-2} , 9.8m]
4. Particles of mass 0.6kg and 0.4kg are connected by a light inextensible string passing over a smooth fixed pulley. Initially both masses are hanging vertically, 30cm above the ground. If the system is released from rest, find the greatest height reached above the ground by the 0.4kg mass. [Ans: 66cm]
5. A particle of mass m_1 lies on a smooth horizontal table and is connected to a freely hanging particle of mass m_2 by a light inextensible string passing over a smooth fixed pulley at the edge of the table. Initially, the system is at rest with m_1 at a distance d from the edge of the table. Show that the time taken for the mass m_1 to reach the edge of the table is $\sqrt{\frac{2d(m_1+m_2)}{m_2 g}}$
6. Particles of masses m_1 and m_2 with $m_2 > m_1$ are connected by a light inextensible string passing over a smooth fixed pulley. Initially, both masses hang vertically with mass m_2 at a height h above the floor. Show that if the system is released from rest, the mass m_2 will hit the floor will hit the floor with a speed $\sqrt{\frac{2(m_2-m_1)gh}{m_1+m_2}}$ and that the mass m_1 will rise a further distance $\frac{(m_2-m_1)h}{m_2+m_1}$ after this occurs.
   Hence or otherwise, show that if $m_1 = 4.5$m and $m_2 = 5.5$m and that m_2 takes 2 seconds before hitting the floor, then; $x = \frac{1}{5}g$
7. Particles of masses m_1 and m_2 with $m_2 > m_1$ are connected by a light inextensible string passing over a smooth fixed pulley. The particles hang vertically and are released from rest. Show that the acceleration of the system is $\frac{(m_2-m_1)g}{m_1+m_2}$ and that the tension in the string is $\frac{2m_1m_2g}{m_1+m_2}$
8. A car of mass 900kg tows a trailer of mass 600kg by means of a rigid tow bar. The car experiences a resistance of 200N and the trailer a resistance of 300N. If the car engine exerts a forward force of 3kN, find the tension in the tow bar and the acceleration of the system. The engine is now switched off and brakes applied to produce a retarding force of 500N. Assuming that the same resistances apply, calculate the retardation of the system. Also comment on the nature and magnitude of the force in the tow bar for this case.
   [Ans: 1300N, $1\frac{2}{3}$ ms^{-2}, $\frac{2}{3}$ ms^{-2}, thrust of 100N]
9. A lorry of mass 3000kg tows a trailer of mass 1000kg along a level road and accelerates uniformly from rest to 18ms^{-1} in 24s. The resistances on the lorry and trailer are proportional to their masses and total to 1200N. Find the;
   (i) Driving force exerted by the engine of the lorry
   (ii) Tension in the tow bar

   [Ans: 4200N, 1050N]

10. A truck A of mass 1000kg pulling a trailer of mass 1000kg pulling a trailer of mass 3000kg. The frictional force on A is 1000N, on B it is 2000N, and the truck engine exerts a force of 8000N. Calculate
    (i) the acceleration of the truck and the trailer
    (ii) The tension in the tow bar connecting A and B

    [Ans: $a = 1.25ms^{-2}$, T = 5750N]

LINEAR MOMENTUM

Momentum of a body is a product of its mass and velocity.

Momentum is a vector quantity and its direction is that of its velocity. When a force, F is applied on a body, it can change the body's velocity form u to v. the size this change depends on the size of the force and the time for which it acts on the body.
From Newton's second law of motion, $F = ma$

but $a = \frac{v-u}{t}$

$$\therefore Ft = mv - mu = change\ in\ momentum$$

The product Ft is called impulse

Impulse is the product of force and the time for which the force acts, and its equal to the change in momentum of the body during that time. It's a vector quantity; whose direction is the same as that of the force. It follows from the equation above the units of impulse are either Ns or $kgms^{-1}$.

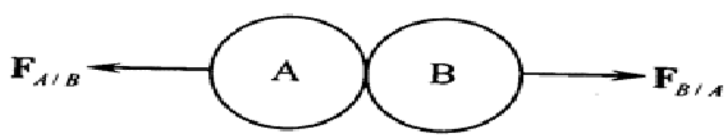
Example
A body of mass 2kg initially moving with a velocity of $1ms^{-1}$ is acted upon by a horizontal force of 6N for 3 seconds. Find the:
(i) Impulse of the given body
(ii) Final speed of the body

Solution
(i) Impulse = Ft = $6 \times 3 = 18Ns$
(ii) $Ft = mv - mu \Rightarrow 18 = 2v - 2 \times 1$
$\therefore v = 10ms^{-1}$

Principal of conservation of momentum

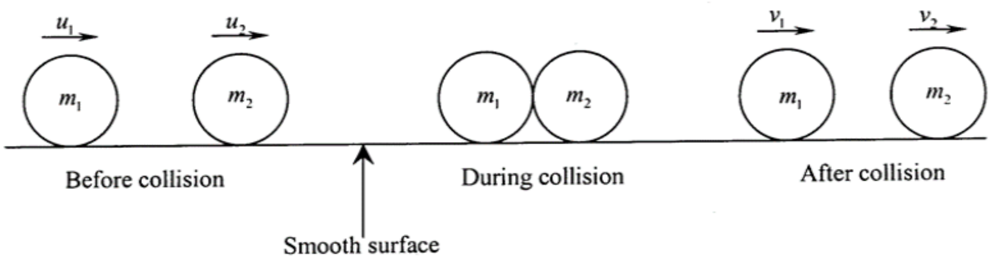
It states that if no external forces act on a system of colliding objects, the total momentum of the objects in a given direction remains constant.

Explanation



Suppose that bodies, A and B, are involved in a collision and that no external forces act on them. From Newton's third law, the force on A due to B, $F_{A/B}$ and the force on B due to A, $F_{B/A}$ are equal and opposite, such that the resultant change in momentum is zero and hence it remains constant.

Proof



From $Ft = mv - mu$,

For the body of mass m_1 : $F_1t = m_1v_1 - m_1u_1$

For the body of mass m_2 : $F_2t = m_2v_2 - m_2u_2$

But the impulse of the bodies are equal but opposite; $\Rightarrow F_1t = -F_2t$

$$\therefore m_1v_1 - m_1u_1 = -(m_2v_2 - m_2u_2)$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Total momentum before collision = total momentum after collision

Therefore the total momentum remains constant

Note:

- If two particles moving in the same straight line collide, there is a short period of contact during which each particle exerts a force on the other. At any instant, the forces exerted by impulse exerted by each body on the other is equal but opposite, since the time taken for action and reaction to act on the particles is negligible.
- It is assumed that during the short period of contact, the effect of any other external forces which act on the particles is negligible

Explanations

1. Explain why a long jumper should normally land on sand

Solution

The force F exerted on a long jumper on coming to rest is given by F = $\frac{change\ in\ momentum}{time\ taken}$.
Since the change in momentum is constant, it implies that if the time taken in coming to rest is increased, then the force exerted on the knees of the jumper reduces. This is what exactly the sand does by enabling the jumper to take more time to come to rest than if he landed on a hard surface. This prevents damage on the legs of the jumper

2. Explain why a goal keeper draws his hands backwards when catching a fast moving hard ball.

   Solution

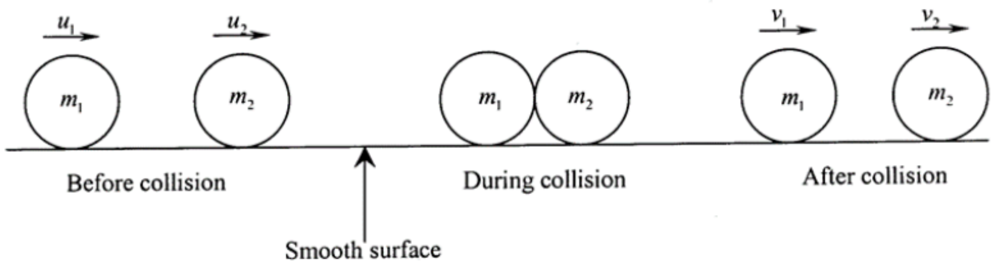
   The goal keeper has to reduce the momentum of the ball to zero and hence draws hands backwards to increase the time to bring the ball to rest and this will in turn reduce the force on the hands. This is not only less painful but also reduces the likelihood of the ball bouncing out of his hands.

3. When a bats man strikes a cricket ball, he follows through in order to keep the bat in contact with the ball for as long time as possible. This increases the impulse and

therefore produces a larger momentum change and so increases the speed at which the ball leaves the bat.

Types of collision

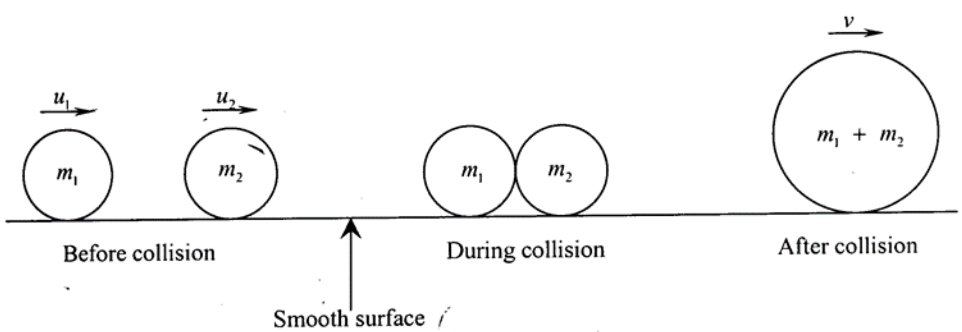
Elastic collision: This is the type of collision where momentum and total kinetic energy are conserved and the particles separate after collision.



i.e $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

and $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Inelastic collision: This is the type of collision where momentum is conserved but total kinetic energy is not conserved. Some of the energy changes to heat and sound which are not recoverable. The particles stick together (coalesce) and move with a common velocity after collision. An inelastic collision in which the two colliding objects stick together and move as a single body after collision is called perfectly inelastic.



$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$
$$\Rightarrow v = \frac{m_1u_1+m_2u_2}{m_1+m_2}$$

Examples

1. For bodies of masses m_1 *and* m_2 with initial speeds u_1*and* u_2 before collision and a common speed v after an inelastic collision, derive an equation for the kinetic energy lost.

Solution

Kinetic energy before collision = $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$

Kinetic energy after collision =$\frac{1}{2}(m_1 + m_2)v^2$ ....(i)

Also $m_1u_1 + m_1u_1 = (m_1 + m_2)v$

$$\Rightarrow v = \frac{m_1u_1+m_2u_2}{m_1+m_2}$$

Substituting for v in equation (i) gives;

$$\text{Kinetic energy after} = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1u_1+m_2u_2}{m_1+m_2}\right)^2$$

$$= \frac{(m_1u_1+m_2u_2)^2}{2(m_1+m_2)}$$

Kinetic energy lost = kinetic energy before collision −kinetic energy after collision

$$\therefore Kinetic\ energy\ lost = \left(\frac{1}{2}m_1{u_1}^2 + \frac{1}{2}m_2{u_2}^2\right) - \left[\frac{(m_1u_1+m_2u_2)^2}{2(m_1+m_2)}\right]$$

$$= \frac{(m_1{u_1}^2+m_2{u_2}^2)}{2} - \frac{(m_1u_1+m_2u_2)^2}{2(m_1+m_2)}$$

$$= \frac{(m_1{u_1}^2+m_2{u_2}^2)(m_1+m_2)-(m_1u_1+m_2u_2)^2}{2(m_1+m_2)}$$

$$= \frac{m_1{u_1}^2+m_1m_2{u_1}^2+m_1m_2{u_2}^2+{m_2}^2{u_2}^2-({m_1}^2{u_1}^2+{m_2}^2{u_2}^2+2m_1m_2u_1u_2)}{2(m_1+m_2)}$$

$$= \frac{m_1m_2{u_1}^2+m_1m_2{u_2}^2-2m_1m_2u_1u_2}{2(m_1+m_2)} = \frac{m_1m_2({u_1}^2+{u_2}^2-2u_1u_2)}{2(m_1+m_2)}$$

$$\therefore kinetic\ energy\ lost = \frac{m_1m_2(u_1-u_2)^2}{2(m_1+m_2)}$$

2. A particle of m_1 moving with a velocity u_1 makes a perfectly elastic collision with a stationary particle of mass m_2. After collision, the first particle moves in the original direction with a velocity v_1 while the second particle moves in the same direction with a velocity v_2. Show that $v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2}$ and $v_2 = \frac{2m_1u_1}{m_1+m_2}$

Solution

From conservation of momentum,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad but\ u_2 = 0$$

$$\Rightarrow m_1(u_1 - v_1) = m_2v_2 \dots\dots\dots (i)$$

From conservation of kinetic energy,

$$\frac{1}{2}m_1{u_1}^2 = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$

$$m_1({u_1}^2 - {v_1}^2) = m_2{v_2}^2$$

$$\Rightarrow m_1(u_1 - v_1)(u_1 + v_1) = m_2{v_2}^2 \dots\dots\dots\text{(ii)}$$

Dividing equations (i) and (ii) gives:

$$u_1 + v_1 = v_2$$

$$\Rightarrow v_1 = v_2 - u_1$$

Substituting for v_2 in equation (i) gives:

$$m_1(u_1 - v_1) = m_2(u_1 + v_1)$$

$$\therefore m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1$$

$$\Rightarrow (m_1 - m_2)u_1 = (m_1 + m_2)v_1$$

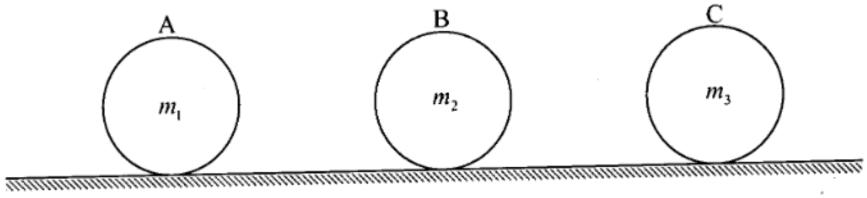
$$\therefore v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2}$$

Also, from (i) $m_1[u_1 - (v_2 - u_1)] = m_2v_2$

$$\Rightarrow m_1u_1 + m_1u_1 - m_1v_2 = m_2v_2$$

$$v_2 = \frac{2m_1u_1}{m_1+m_2}$$

3. Balls A, B and C of masses m_1, m_2 and m_3 respectively lie on a horizontal surface as shown below.



The balls are initially at rest. Ball A is projected with a velocity v_1 and moves towards B making a perfectly elastic collision with B. If B moves and makes a perfectly inelastic collision with C, show that both B and C will move with a velocity,

$$v_3 = \frac{2m_1m_2v_1}{(m_1+m_2)(m_2+m_3)}$$

Solution

Let v_2 be velocity of A after collision with B and v_B be the velocity of B after collision with A

Consider A and B (elastic collision)

Conservation of momentum: $m_1v_1 = m_1v_2 + m_2v_B$

$$\Rightarrow v_2 = \frac{m_1v_1 - m_2v_B}{m_1} \text{ ....(i)}$$

Conservation of kinetic energy:

$$\frac{1}{2}m_1{v_1}^2 = \frac{1}{2}m_1{v_2}^2 + \frac{1}{2}m_2{v_B}^2$$

$$\Rightarrow m_1{v_1}^2 = m_1{v_2}^2 + m_2{v_B}^2 \text{ ....(ii)}$$

Substituting for v_2 in equation (ii) gives;

$$m_1{v_1}^2 = m_1\left(\frac{m_1v_1 - m_2v_B}{m_1}\right)^2 + m_2{v_B}^2$$

$$m_1{v_1}^2 = (m_1v_1 - m_2v_B)^2 + m_1m_2{v_B}^2$$

$$\Rightarrow 2m_1m_2v_1v_B = {m_2}^2{v_B}^2 + m_1m_2{v_B}^2$$

$$\therefore 2m_1v_1 = (m_1+m_2)v_B$$

$$\Rightarrow v_B = \frac{2m_1v_1}{m_1+m_2}$$

Consider bodies B and C (inelastic collision)

$$m_2v_B + 0 = (m_2+m_3)v_3$$

Substituting v_B gives:

$$m_2\frac{2m_1v_1}{m_1+m_2} = (m_2+m_3)v_3$$

$$\frac{2m_1m_2v_1}{m_1+m_2} = (m_2+m_3)v_3$$

$$\Rightarrow v_3 = \frac{2m_1m_2v_1}{(m_1+m_2)(m_2+m_3)}$$

4. A particle P of mass m_1 moving with a speed u_1 collides head on with a stationary particle of mass m_2. If the collision is elastic and the speeds of the particles after impact are v_1 and v_2, show that for $\phi = \frac{m_2}{m_1}$, then

(i) $\frac{v_2}{u_1} = \frac{2}{1+\phi}$ (ii) Q gains $\frac{4\phi}{(1+\phi)^2}$ of the total energy of the system

Solution

(i) $m_1u_1 = m_1v_1 + m_2v_2$

$\Rightarrow m_1(u_1 - v_1) = m_2v_2$ ……………(i)

From conservation of kinetic energy,

$$\frac{1}{2}m_1{u_1}^2 = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$

$m_1(u_1 - v_1)(u_1 + v_1) = m_2{v_2}^2$ ……………(ii)

Dividing equations (i) and (ii) gives:

$$u_1 + v_1 = v_2$$

$$\Rightarrow v_1 = v_2 - u_1$$

Substituting for v_1 in equation (i) gives

$$m_1[u_1 - (v_2 - u_1)] = m_2v_2$$

$$\Rightarrow 2u_1 - v_2 = \frac{m_2}{m_1}v_2$$

$$\therefore 2u_1 = (1 + \phi)v_2$$

$$\Rightarrow \frac{v_2}{u_1} = \frac{2}{1+\phi}$$

(ii) Initial energy $= \frac{1}{2}m_1{u_1}^2 + 0 = \frac{1}{2}m_1{u_1}^2$

Energy of Q after collision $= \frac{1}{2}m_2{v_1}^2$

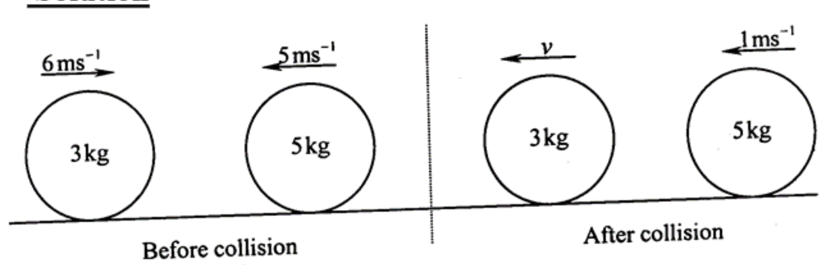
Fractional gain in energy $= \frac{\frac{1}{2}m_1{u_1}^2}{\frac{1}{2}m_2{v_1}^2} = \frac{m_1{u_1}^2}{m_2{v_1}^2} = \frac{m_2}{m_1}\left(\frac{v_2}{u_1}\right)^2$

But $\frac{m_2}{m_1} = \phi$ $\quad and \quad \frac{v_2}{u_1} = \frac{2}{1+\phi}$

Therefore fractional change in energy $= \phi\left(\frac{2}{1+\phi}\right)^2 = \frac{4\phi}{(1+\phi)^2}$

5. Two bodies of masses 3kg and 5kg traveling in opposite directions on a horizontal surface collide. The velocities of the 3kg and 5kg bodies before collision are 6ms^{-1} and 5ms^{-1} respectively. They separate after collision and move in the same direction in which the 5kg block was moving before collision. Given that the velocity of the 5kg body after collision is 1ms^{-1}, find the speed of the 3kg body impact and the percentage loss in energy if there is any (if at all it is there)

Solution



From conservation of momentum, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Since momentum is a vector quantity (taking directions of velocity), we have to take into account the directions of velocity given

Considering all velocities to the right (→) to positive:

$$(3 \times 6) + 5 \times (-5) = 3 \times (-v) + 5 \times (-1)$$

$\Rightarrow v = \frac{2}{3}$ms^{-1}

Kinetic energy before collision $= \frac{1}{2}m_1{u_1}^2 + \frac{1}{2}m_2{u_2}^2$

$= \frac{1}{2} \times 3 \times 6^2 + \frac{1}{2} \times 5 \times 5^2 = 116.5\, Joules$

Kinetic energy after collision $= \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$

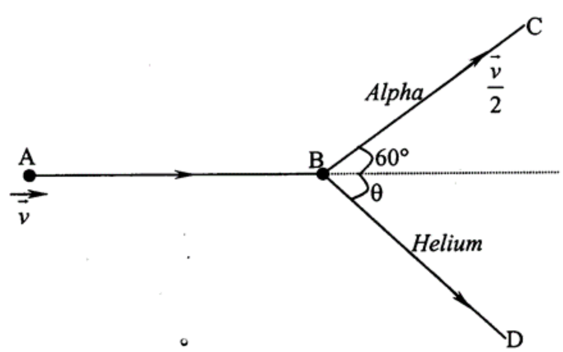
$= \frac{1}{2} \times 3 \times \left(\frac{2}{3}\right)^2 + \frac{1}{2} \times 5 \times (-1)^2 = 3.2\, Joules$

Loss in energy = Kinetic energy before – kinetic energy after collision

= 116.5 – 3.2 = 113.3

Percentage loss in energy $= \frac{113.3}{116.5} \times 100 = 97.3\%$

6.



An alpha particle A of mass 4 units is incident with a velocity v on a stationary helium nucleus B of mass 4 units. After collision, A moves in direction BC with a velocity $\frac{v}{2}$, where BC makes an angle of 60^0 with the initial direction AB. The helium nucleus moves along BD. Calculate the velocity of rebound of helium nucleus along BD and the angle it makes with the direction AB.

Solution

Let x be the velocity of helium after impact.

Consider the conservation of horizontal momentum:

$4v + 0 = 4\left(\frac{v}{2}\cos 60\right) + 4(x\cos\theta) \ldots\ldots (i)$

Consider the conservation of vertical momentum

$0 + 0 = 4\left(\frac{v}{2}\sin 60\right) + 4(x\sin\ \theta)$…………(ii)

From (i) $v = \frac{v}{2}\cos 60 + x\cos\theta$

$v = \frac{v}{4} + x\cos\theta$

$\therefore \cos\theta = \frac{3v}{4x}$ ……(iii)

From (ii) $4x\sin\theta = 4 \times \frac{v}{2}\sin 60$

$\Rightarrow x\sin\theta = \frac{\sqrt{3}v}{4}$

$\therefore \sin\theta = \frac{\sqrt{3}v}{4x}$

But $cos^2\theta + sin^2\theta = 1$

$$\therefore \left(\frac{3v}{4x}\right)^2 + \left(\frac{\sqrt{3}v}{4x}\right)^2 = 1$$

$$\frac{9v^2}{16x^2} + \frac{3v^2}{16x^2} = 1$$

$$\Rightarrow \frac{12v^2}{16x^2} = 1$$

$$\therefore x = \sqrt{\frac{12v^2}{16}} = \frac{\sqrt{3}}{2}v$$

Therefore the velocity of helium after collision is $\frac{\sqrt{3}}{2}v$

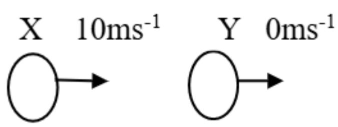
From $\cos\theta = \frac{3v}{4x}$ *and* $x = \frac{\sqrt{3}}{2}v$

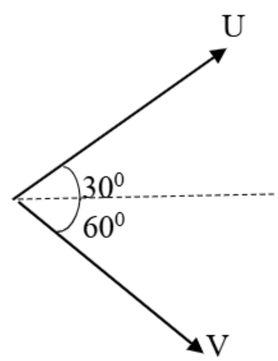
$$\cos\theta = \frac{3v}{4\times\frac{\sqrt{3}}{2}v} = \frac{3\times2}{4\times\sqrt{3}}$$

$$\therefore \cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \theta = 30^0$$

7. An object X of mass M, moving with a velocity of $10ms^{-1}$ collides with a stationary object Y of equal mass. After collision, X moves with a speed U, at an angle of 30^0 to its initial direction, while Y moves with a speed V at an angle of 90^0 to the new direction.
   (i) Calculate the speeds U and V
   (ii) Determine whether the collision is elastic or not

Solution





(i) Applying the conservation of horizontal momentum;

$$M \times 10 = MU\cos 30 + MV\cos 60$$

$$10 = U\frac{\sqrt{3}}{2} + \frac{V}{2}$$

$$\Rightarrow 20 = U\sqrt{3} + V \text{.............(i)}$$

Applying the conservation of vertical momentum;

$$M \times 0 = MU\sin 30 - MV\sin 60$$

$$0 = \frac{U}{2} - V\frac{\sqrt{3}}{2}$$

$$\Rightarrow U = V\sqrt{3} \ldots\ldots\ldots\ldots..(ii)$$

Substituting for U in equation (i)

$$\Rightarrow 20 = 3V + V$$

$$\therefore V = 5ms^{-1}$$

From equation (ii) U = $5 \times \sqrt{3} = 8.66ms^{-1}$

(ii)

Total kinetic energy before collision = $\frac{1}{2} \times M \times 10^2 = 50M\ J$

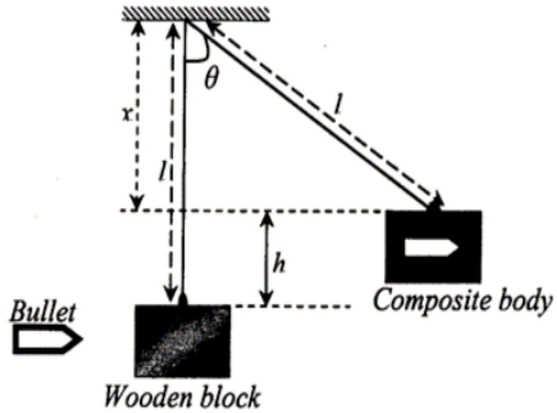
Total kinetic energy after collision = $\frac{1}{2}M \times 5^2 + \frac{1}{2}M(5\sqrt{3})^2 = \frac{25M}{2} + \frac{75M}{2} = 50M\ J$

Kinetic energy is conserved hence it is an elastic collision

Trial questions

1. A ball A of mass 10kg moving with a speed of 8ms^{-1} collides with another ball B of mass 20kg initially at rest. After the collision, A and B move in directions making angles of 30^0 and 45^0 respectively with the initial direction of motion of A. Calculate the speeds of A and B after the collision. [Ans: 5.85ms^{-1}, 2.07ms^{-1}]
2. Two balls P and Q of masses m_1 and m_2 respectively lie on a smooth surface in a straight line. If the balls are projected with velocities u_1 and u_2 respectively in the same direction, show that on colliding elastically, then P will move with a velocity v_1 given by;
$$v_1 = \frac{(m_1{}^2 - m_2{}^2)u_1 + 2m_1m_2(u_2 - u_1) + 2m_2{}^2(u_1 - u_2)}{m_1{}^2 - m_2{}^2}$$
3. A body of mass m_1 traveling with a speed of u_1 makes a head on collision with a stationary particle B of mass m_2. If the collision is elastic, and the speeds A and B after collision is elastic, and the speeds of A and B after collision are v_1 and v_2 respectively, show that for $X = \frac{m_1}{m_2}$, then;
(i) $\frac{u_1}{v_1} = \frac{X+1}{X-1}$ (iii) $\frac{v_2}{v_1} = \frac{2X}{X-1}$
(ii) $X = \left(\frac{v_1+u_1}{u_1-v_1}\right)$
4. A body explodes and produces two fragments of masses m and M. If the velocities of the fragments are u and v respectively. Show that the ratio of the kinetic energies of the fragments is $\frac{E_1}{E_2} = \frac{M}{m}$ where E_1 is the kinetic energy of m and E_2 is the kinetic energy of M.
5. A car of mass 1000kg traveling at a uniform velocity of 20ms^{-1} collides perfectly inelastic ally with a stationary car of mass 1500kg. Calculate the loss in kinetic energy as the result of the collision. [Ans: 1.68×10^5 J]
6. A snooker ball X of mass 0.3kg moving with a velocity 5ms^{-1}, hits a stationary ball Y of mass 0.4kg. Y moves off with a velocity of 2ms^{-1} at 30^0 to the initial direction of X and X moves at an angle θ to its initial direction. Find the velocity of X and the value of θ
[Ans: 3ms^{-1} , θ = 27^0]

Ballistic Pendulum



The ballistic pendulum is a device used to determine the speed of a bullet. Consider a bullet of mass m traveling horizontally, and incident with a velocity u ms^{-1} on a stationery wooden block of mass M, suspended by a light vertical string of length ,*l.* If the composite body (the block and the bullet), moves with a velocity v after collision, and given that the composite body comes to rest a height h above the original position of the wooden block,

From conservation of momentum,

$$mu + 0 = (m + M)v$$

$$\Rightarrow u = \frac{(m+M)v}{m} \text{ ....(i)}$$

From conservation of energy,

K.e of composite body just after collision = P.E of composite body at a height h

$$\frac{1}{2}(m + M)v^2 = (m + M)gh$$

$$\Rightarrow v = \sqrt{2gh} \text{ ....(ii)}$$

Substituting for v in equation (i) gives;

$$u = \frac{(m+M)\sqrt{2gh}}{m}$$

Also from the diagram,

$$x + h = l \Rightarrow h = l - x$$

But $\cos\theta = \frac{x}{l} \Rightarrow x = l\cos\theta$

$$h = l - l\cos\theta = l(1 - \cos\theta)$$

From equation (ii) $v^2 = 2gh$

Substituting for h gives, $v^2 = 2gl(1 - \cos\theta)$

$$\therefore \theta = \cos^{-1}\left(1 - \frac{v^2}{2gl}\right)$$

Therefore, depending on a particular question scenario, any of the variables can be calculated.

Example

A bullet of mass 20g traveling at a speed of 200ms^{-1} embeds itself in a block of mass 980g suspended by a string, such that it swings freely. Find:

(i) The vertical height through which the block rises,

(ii) How much of the bullet's energy becomes internal energy

Solution

$u = 200, v = ?\ ,\ m = 20g = 0.02kg, M = 980g = 0.98kg$

(i) From conservation of momentum,

$$mu = (m+M)v$$
$$\Rightarrow 0.02 \times 200 = (0.98 + 0.02)v$$
$$v = 4\text{ms}^{-1}$$

From conservation of energy;

$$\frac{1}{2}(m+M)v^2 = (m+M)gh$$
$$\Rightarrow \frac{1}{2} \times 4^2 = 9.81 \times h$$
$$h = 0.82m$$

(ii) KE before collision $= \frac{1}{2}mu^2 + \frac{1}{2}M \times 0 = \frac{1}{2} \times 0.02 \times 200^2 = 400\,Joules$

KE after collision $= \frac{1}{2}(m+M)v^2 = \frac{1}{2} \times 1 \times 4^2 = 8\,Joules$

The difference in energy becomes internal energy

$$\Rightarrow internal\ energy = 400 - 8 = 392\,Joules$$

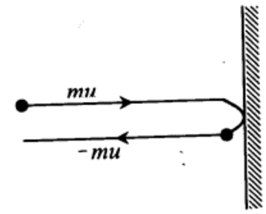
Trial questions

1. A bullet of mass 20g is fired horizontally with a speed of 200ms^{-1} into a 2kg block of wood suspended by a string 1m long. The bullet is embedded in the block. The bullet is embedded in the block. Calculate the maximum inclination of the string to the vertical

   [Ans: 37^0]
2. A bullet of mass 10g travelling horizontally at a speed of 100ms^{-1} strikes a block of wood of mass 900g suspended by a light vertical string and is embedded in the block which subsequently swings freely. Find the
   (i) Vertical height through which the block rises
   (ii) Kinetic energy lost by the bullet

   [Ans: 6.2 ×10^{-2} m , 49.99J]
3. A bullet of mass 40g is fired from a gun at 200ms^{-1} and hits the block of mass 2kg which is suspended by a light vertical string 2m long. If the bullet gets embedded in the wooden block. Calculate the maximum angle the string makes with the vertical and state the factors on which the angle of swing depends [Ans:52.4^0]
4. A bullet of mass 20g travelling horizontally at 100ms^{-1} embeds itself in the centre of a block of wood of mass 1kg which is suspended by a light vertical light 1m in length. Calculate the maximum inclination of the string to the vertical [Ans: 37^0]

Force on a surface



When a particle of mass m is traveling horizontally at a speed,u strikes a wall, it rebounds with an equal but opposite momentum.

Change in momentum = $mu - (-mu)$

= 2mu

But impulse = change in momentum

$\Rightarrow Ft = 2mu$

$\therefore F = \frac{2mu}{t}$

Where t is the time for which the body is in contact with the surface

Consider a jet of water striking a surface and doesnot rebound. Here, the surface is said to have destroyed the momentum of the water. This loss in momentum is equal to the impulse of the force exerted on the water by the surface.

If the amount of water hitting the surface per second is known, then momentum destroyed per second can be calculated.

Impulse = change in momentum

$\Rightarrow Ft = momentum\ destroyed$

If t = 1s

$F \times 1 = momentum\ destroyed\ per\ second$

$\therefore F = Momentum\ destroyed\ per\ second$

Examples

1. Water issues horizontal from a horse of cross sectional area 5cm^2 at a speed of 10ms^{-1}. The water strikes a vertical wall with out rebounding. Find the magnitude of the force acting on the wall if the density of water is 1000kgm^{-3}.

Solution

A = 5cm^2 = 0.0005m$_2$, v = 10ms^{-1}, $\rho = $ 1000kgm^{-3}, F = ?

$F = Momentum\ destroyed\ per\ second$

But $Momentum = mass\ \times velocity$

$\Rightarrow Momentum\ per\ second = (mass\ per\ second) \times (velocity) \ldots\ldots (i)$

Also $mass\ per\ second = (volume\ per\ second) \times (density) \ldots \ldots \ldots (iii)$

But $volume = Area\ \times height$

$\Rightarrow volume\ per\ second = Area\ \times height\ per\ second$

But height per second = velocity

$\Rightarrow volume\ per\ second = Area \times velocity$

$= 0.0005 \times 10 = 0005$m^{3}s^{-1}

Substituting for volume per second in equation (iii) gives;

Mass per second = $0.0005 \times 1000 = 5$kgs^{-1}

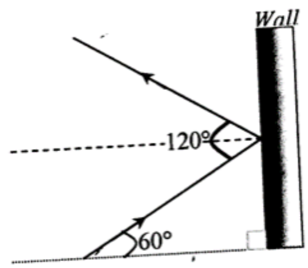
Substituting for mass per second in equation (ii) gives

Momentum per second = $5\times 10 = 50$kgms^{-2}

Therefore from equation (i) Force = 50N

2. A ball of mass 500g, is traveling at a speed of 10ms^{-1} at 60^0 to the horizontal strikes a vertical wall and rebounds with the same speed, at 120^0 to its initial direction. If the ball is in contact with the wall for 8×10^{-3}s. Calculate the average force exerted on the wall by the ball.

Solution



Considering the horizontal component of motion, and taking the velocity towards the wall as positive;

$$v = 10\cos 60 = 10 \times 0.5 = 5\,ms^{-1}$$

This implies that horizontal velocity of the ball after impact with the wall is -5ms^{-1}

From momentum = $mass \times velocity$

Momentum before impact = $0.5 \times 5 = 2.5$

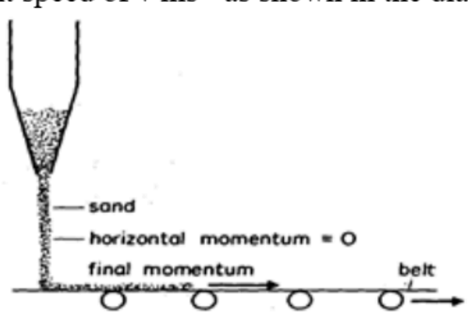
Momentum before impact = $0.5 \times -5 = -2.5$

Change in momentum = 2.5 − -2.5 = 5kgms^{-1}

But F = $\frac{change\ in\ momentum}{time\ taken} = \frac{5}{0.008} = 625N$

Conveyer belt

Suppose a substance say sugar/sand falls from a pipe vertically onto a horizontal conveyer belt moving forward at a constant speed of v ms^{-1} as shown in the diagram below



As the sugar strikes the belt, its horizontal velocity is zero hence horizontal momentum equals zero. If sugar is falling at a rate of R kgs^{-1}, in time, t seconds, the mass on the belt is Rt kg.
On the belt, the substance moves with a velocity v ms^{-1} so it gains a velocity v in 1 second. This means that every second, a substance of mass R kg is accelerated from rest to a speed v

After time t, momentum = Rtv $kgms^{-1}$

Change in momentum = Rtv – 0 = Rtv

The rate of change of momentum = $\frac{Rtv}{t} = Rv$

But from Newton's second law; Force = rate of change of momentum

$$\Rightarrow F = Rv$$

Therefore the force experienced on the conveyer belt is equal to the product of the belt speed and the rate at which the mass is added.

Example

Sugar falls onto a conveyer belt at a constant rate of $100gs^{-1}$ and the belt maintains a horizontal velocity of $5cms^{-1}$. Find the force necessary to maintain this constant speed of the belt.

Solution

Rate R = $100gs^{-1} = 0.1kgs^{-1}$

Velocity of the belt = $0.05ms^{-1}$

But $F = Rv$

$= 0.1 \times 0.05 = 5 \times 10^{-3} N$

Recoil velocity of a gun

If a bullet of mass m is fired from a rifle of mass M with a velocity of v ms-1. Initially, the total momentum of the bullet and rifle is zero.

From the principle of conservation of linear momentum, when the bullet is fired, the total momentum of the bullet and rifle is still zero, since no external force has acted on them.

So if V is the recoil velocity of the rifle, then

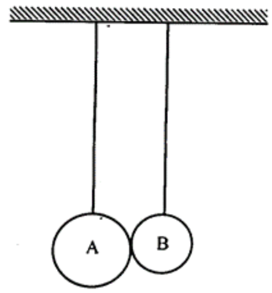
$$mv(bullet) + MV(rifle) = 0$$

$$\therefore MV = -mv$$

The momentum of the rifle is equal and opposite to that of the bullet, thus $V = \frac{m}{M}v$ where V is the recoil velocity of the rifle in the opposite direction of that of the bullet.

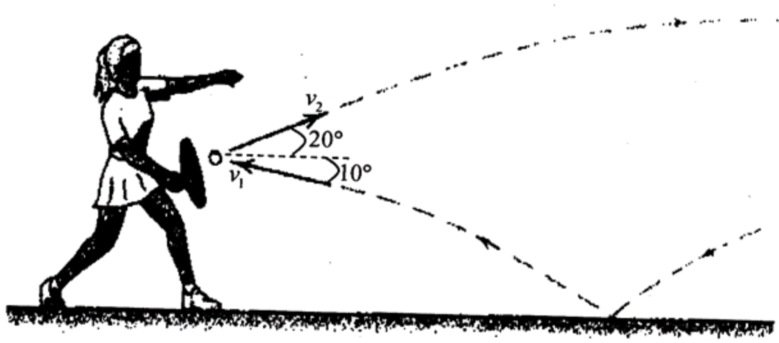
Trial questions

1. Jinja factory waste pipe emits fluid of density $1200kgm^{-3}$ through a pipe of cross-sectional area 25×10^{-4} m^2. The fluid leaves the pipe horizontally with a speed of $8ms^{-1}$ and strikes a vertical wall without rebounding. Find the magnitude of the force acting on the wall. [Ans: 192N]
2. Two pendula of equal length 4m have bobs A and B of masses 3M and M respectively. The pendula are hung with the bobs in contact as shown in the figure.



Bob A is displaced such that the string on to which its attached makes an angle 60^0 with the vertical, and released. If A makes a perfectly inelastic collision with B, find the height to which B rises. [Ans: 1.13m]

3. A ball of mass 0.2kg falls from a height of 45m. On striking the ground, it rebounds in 0.1s with two-thirds of the velocity with which it struck the ground. Calculate
   (i) Momentum change on hitting the ground
   (ii) The force on the ball due to the impact [Ans: 10 Ns , 100N]
4. A ball of mass 0.05kg strikes a smooth vertical wall normaly four times in 2 seconds with a velocity of $10ms^{-1}$. Each time the ball rebounds with the same speed of $10ms^{-1}$. Calculate the average force exerted on the wall. [Ans: 2.0N]
5. A hose directs a horizontal jet of water moving with a velocity of $20ms^{-1}$, onto a vertical wall. The cross sectional area of the jet is 5×10^{-4} m^2. If the density of water is 1000 kgm^{-3}, calculate the force on the wall assuming the water is brought to rest there.
   [Ans: 200N]
6. A tennis player strikes the tennis ball with her racket while the ball is still rising. The ball's speed before impact with the racket $v_1 = 15ms^{-1}$ and after impact its speed is $v_2 = 22ms^{-1}$, with the shown directions



If the 60-g ball is in contact with the racket for 0.05s, determine the;
   (i) magnitude of the average force exerted by the racket on the ball
   (ii) angle made by the resultant with the horizontal
   [Ans: 43.0N, 8.68^0]

CHAPTER 4: SOLID FRICTION

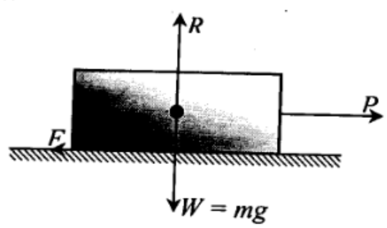
Friction is the force that opposes the relative motion of two surfaces in contact. The direction of functional force is always opposite to the direction of motion of the body. Bodies which can move across each other without encountering resistance are said to be smooth, while those whose relative motion is resisted are said to be rough. Solid friction can either be static or kinetic friction

Static friction: This is the force that opposes the relative motion of two surfaces in contact which are at rest but have a tendency to start moving. It prevents motion

Kinetic friction: this is the force that opposes relative motion of two surfaces, which are already in motion. Kinetic friction is also called dynamic or sliding friction. It slows down motion

Mechanism of friction

Consider a body of mass, m resting on a rough surface and that a small force P is applied on a body as shown below



For small values of the applied force P, it is observed that the body does not move. This implies that R = mg and P = F i.e the body is in equilibrium

Experiments show that even when P is gradually increased, motion does not take place until a particular value of P is reached. This means that the frictional force between two surfaces increases as the tendency to move increases, up to a limiting value. When the limiting frictional force is reached, the body is at a point of starting to move (Point of impending motion)

Therefore, **limiting friction** is the maximum force that opposes relative motion of two surfaces in contact, if the two surfaces were not in motion originally.

Laws of solid friction

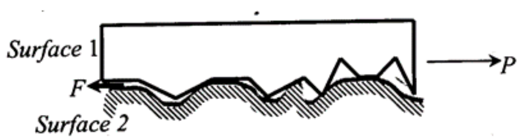
(i) Friction acts parallel to the surfaces in contact, and in a direction so as to oppose their relative motion

(ii) The frictional force does not depend on the area of contact of the given surfaces provided that the normal reaction is constant, but depends on the nature of the surfaces in contact.

(iii) (a) When the surfaces are at rest, the limiting frictional force (static friction) is directly proportional to the normal reaction R
i.e $F \propto R \Rightarrow F = \mu_s R$ where μ_s is the coefficient of static friction
(b) When motion occurs, the kinetic friction is directly proportional to the normal reaction but does not depend on the relative velocity between the two surfaces
i.e $F \propto R \Rightarrow F = \mu_k R$ where μ_k is the coefficient of kinetic friction

Molecular theory explanation of solid friction

Friction is a surface phenomenon. Consider two rough surfaces in contact as shown below

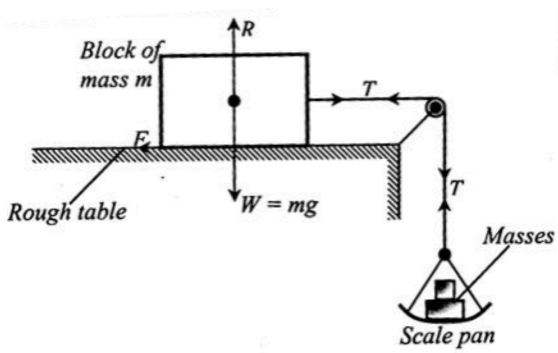


Experimental evidence shows that the area of contact of the two surfaces is very small. Therefore the pressure exerted on the surfaces is very high. As a result, molecules of the surfaces are pushed into close proximity, hence forming welds on welded joints. This creates some degree of interlocking between these irregular projections of the surfaces. Therefore these welds have to be broken before motion can take place, hence an opposing force is developed (This explains why solid friction is maximum just before motion takes place). This explains the first law.

When the area of contact is reduced, the number of contact points remain approximately the same, and hence the resisting force is not affected, which explains why friction is independent of area. This explains the second law.

An increase in weight (and hence normal reaction) of the upper surface increases the pressure at the welded joints. This leads to a greater degree of interlocking, and hence a bigger force is required to cause motion. This explains why friction is dependent on (directly proportional to) normal reaction, and hence explains the third law.

Experiment to determine the coefficient of static friction



Procedure

Known masses are added onto the scale pan in bits until the block of known mass m is just about to slide, when the maximum frictional force is reached. The total mass, M of the scale pan together the added masses is determined.

Either:

At the point of impending motion,

Limiting frictional force, F = weight of the scale pan together with the added masses

$$\mu_s R = Mg \quad \text{but R = mg}$$

$$\therefore \ \mu_s = \frac{Mg}{mg}$$

$$\Rightarrow \mu_s = \frac{M}{m}$$

$$\mu = \frac{weight\ of\ scale\ pan\ together\ with\ added\ masses}{weght\ of\ wooden\ block}$$

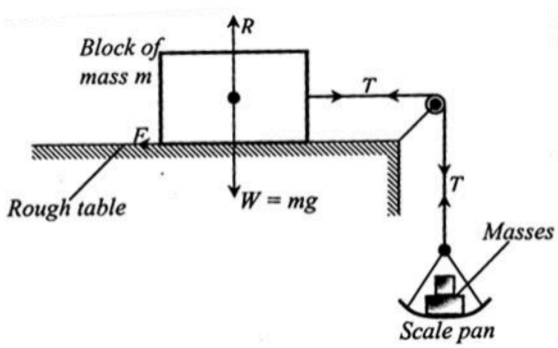
Alternatively:

The process is repeated for different values of m (obtained by adding known masses to the block), and the corresponding values of M determined.

A graph M against m is plotted, and it's a straight line through the origin, whose slope gives the value of the coefficient of static friction, μ_s

Note: in this experiment, one of the given alternatives is enough for a particular experiment.

Experiment to determine the coefficient of dynamic/kinetic/sliding friction



Procedure

Known masses are added to the scale pan, and each time a mass is added, the block of mass m is given a slight push. A certain time comes when the block continues to move with a constant velocity after being pushed. The mass M of the scale pan together with its contents is determined.

Either:

For motion,

Kinetic friction, F = weight of the scale pan together with the added masses

$$\mu_k R = Mg \quad \text{but R = mg}$$

$$\therefore\ \mu_k = \frac{Mg}{mg}$$

$$\Rightarrow \mu_k = \frac{M}{m}$$

$$\mu = \frac{weight\ of\ scale\ pan\ together\ with\ added\ masses}{weght\ of\ wooden\ block}$$

Alternatively:

The process is repeated for different values of m (obtained by adding known masses to the block), and the corresponding values of M determined.

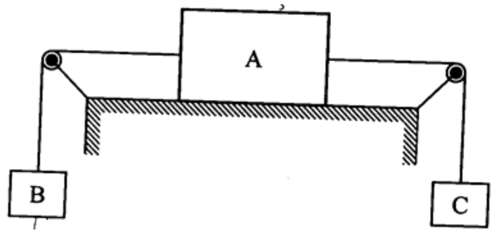
A graph M against m is plotted, and it's a straight line through the origin, whose slope gives the value of the coefficient of kinetic friction, μ_k

The reader should note that much as the setups for the two experiments are the same, the procedures are different.

The reader can also determine the coefficients using an inclined plane by reading the reference books.

Examples

1.

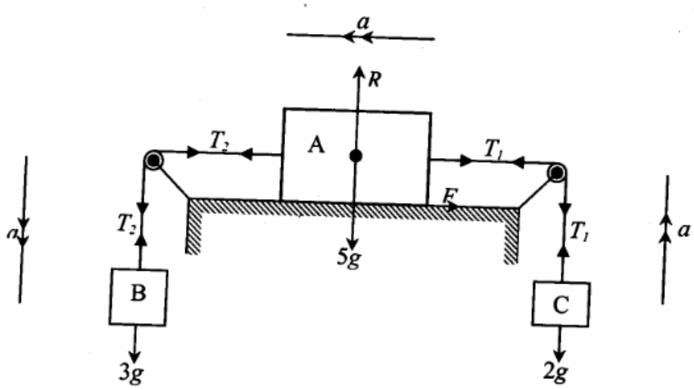


Three blocks A, B and C of masses 5kg, 3kg and 2kg respectively are arranged as shown above, and the system is released from rest. If the coefficient of sliding friction between the block A and the table is $\frac{1}{7}$, calculate;

(i) The acceleration of the system

(ii) The tension in the strings

Solution



(i) $F = \mu_k R$ *but* $\mu_k = \frac{1}{7}$ *and* $R = 5\text{g}$

$\Rightarrow F = \frac{1}{7} \times 5\text{g} = \frac{5}{7}\text{g}$

Block A:

Net force = $T_2 - T_1 - F$

$\Rightarrow 5a = T_2 - T_1 - \frac{5}{7}\text{g}$………(i)

Block B:

Net force = $3\text{g} - T_2$

$\Rightarrow 3a = 3\text{g} - T_2$ …………..(ii)

Block C:

Net force = $T_1 - 2\text{g}$

$\Rightarrow 2a = T_1 - 2\text{g}$ ………..(iii)

Adding the three equations ie (i) + (ii) + (iii) gives;

$$5a + 3a + 2a = T_2 - T - \frac{5}{7}\text{g} + 3\text{g} - T_2 + T_1 - 2\text{g}$$

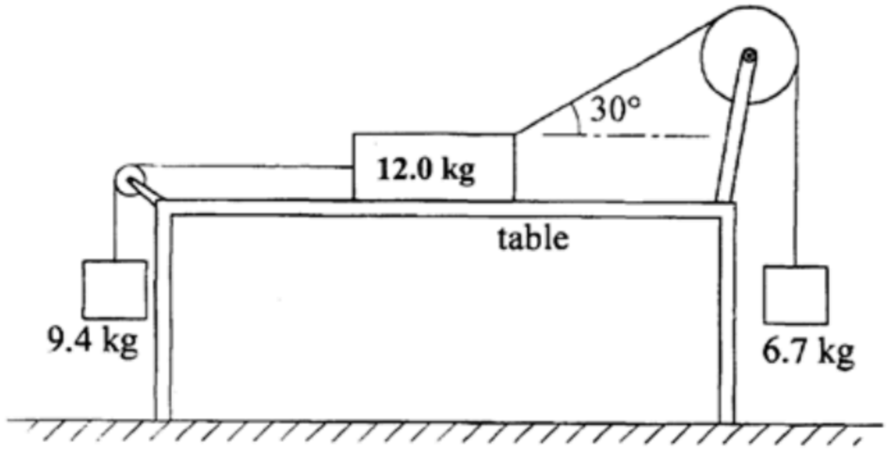
$$\Rightarrow 10a = \frac{2}{7}\text{g} \quad \therefore a = \frac{2}{70}\text{g} = 0.28\text{ms}^{-2}$$

(ii) Substituting for a in equations (ii) and (iii) gives;

$$3 \times 0.28 = 3 \times 9.81 - T_2 \Rightarrow T_2 = 28.6N$$

$$2 \times 0.28 = T_1 - 2 \times 9.81 \Rightarrow T_1 = 20.18N$$

2. The diagram below shows three masses connected by inextensible strings which pass over smooth pulleys. The coefficient of friction between the table and the 12.0 kg mass is 0.25

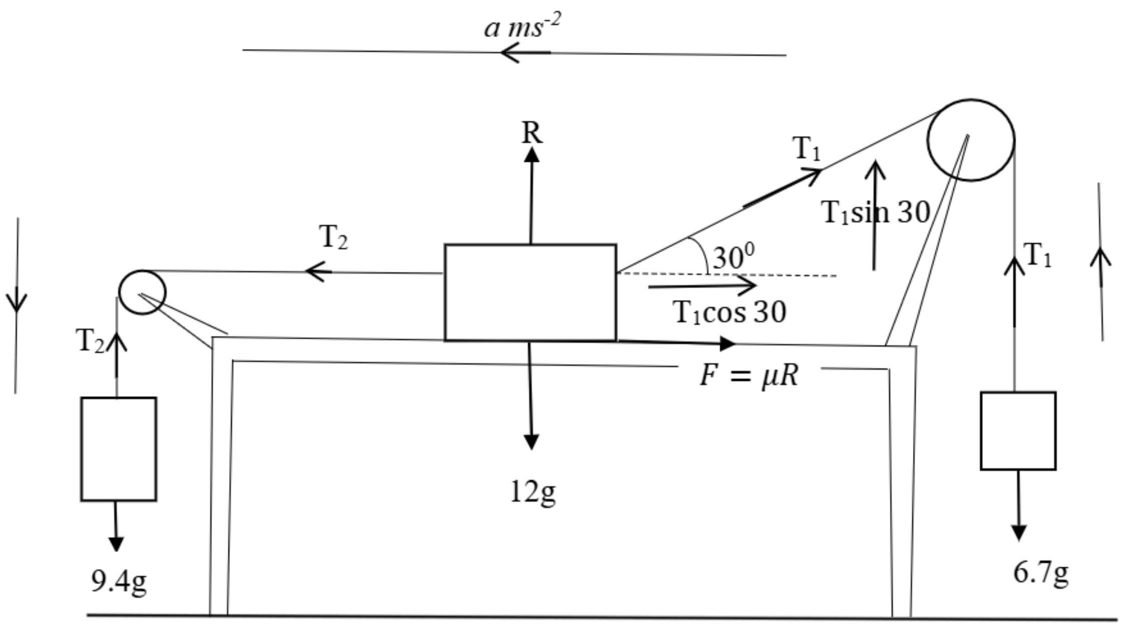


If the system is released from rest, determine the

(i) Acceleration of the 12.0kg mass

(ii) Tension in each string

Solution



(i) Vertically when the body is at rest:

$$R + T_1 \sin 30 = 12g$$

$$R + \frac{T_1}{2} = 12g \quad \text{since } \sin 30 = \frac{1}{2}$$

$$\Rightarrow R = 12g - \frac{T_1}{2}$$

But $F = \mu R \;\; where\; \mu = 0.25 = \frac{1}{4}$

$$\Rightarrow F = \frac{1}{4}\left(12g - \frac{T_1}{2}\right) = 3g - \frac{T_1}{8}$$

Considering the 12kg mass;

$T_2 - T_1 \cos 30 - F = 12a$

$T_2 - T_1 \frac{\sqrt{3}}{2} - \left(3g - \frac{T_1}{8}\right) = 12a$

$T_2 - \frac{\sqrt{3}}{2} T_1 - 3g + \frac{T_1}{8} = 12a$

$T_2 - 3g + T_1 \left(\frac{1}{8} - \frac{\sqrt{3}}{2}\right) = 12a$

$T_2 - 0.74T_1 - 3g = 12a$ ……………(i)

Considering the 9.4 kg mass;

$9.4 - T_2 = 9.4a$ ………………..(ii)

Considering the 6.7 kg mass;

$T_1 - 6.7g = 6.7a$ ……………(iii)

Adding equation (i) and (ii) gives;

$6.4g - 0.74T_1 = 21.4a$ ……………..(*)

Multiplying equation (iii) by 0.74 gives;

$0.74T_1 - 4.958g = 4.958a$ ………………..(**)

Adding equation (*) and (**) gives;

$1.442g = 26.358a$

$a = \frac{1.442 \times 9.81}{26.358} = 0.537$ ms^{-2}

(ii)

From $T_1 - 6.7g = 6.7a$

$T_1 = 6.7 \times 0.535 + 6.7 \times 9.81 = 69.32N$

From $9.4 - T_2 = 9.4a$

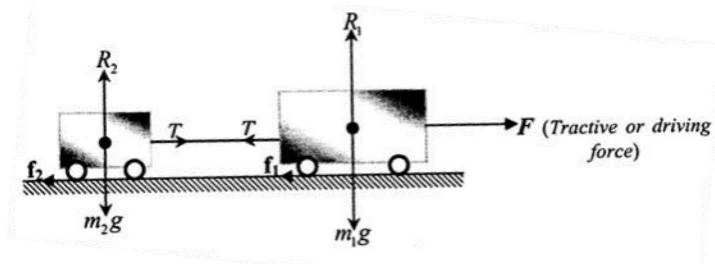
$T_2 = 9.4 \times 9.81 - 9.4 \times 0.537 = 87.17N$

3. A tractor of mass 2.0×10^3 kg is used to pull a car of mass 1.0×10^3 kg to which it's connected by a chain whose mass is negligible. The tractor pulling steadily moves the car from rest along a horizontal road through a distance of 12.5m in 5s. The coefficient of kinetic friction between the tyres of the tractor and the road is 0.4, while that between the tyres of the car and the road is 0.2. Find the;

(i) Tractive pull exerted by the tractor's engine

(ii) Power developed by the tractor's engine

Solution



(i) From the information given,

$m_1 = 2\times10^3$ kg, $m_2 = 1\times103$ kg, $\mu_1 = 0.4, \mu_2 = 0.2, u = 0, s = 12.5m, t = 5s$

From $S = ut + \frac{1}{2}at^2$, $12.5 = 0 \times 5 + \frac{1}{2} \times a \times 5^2 \Rightarrow a = 1ms^{-2}$

Also $f_1 = \mu_1 \times R_1$ but $R_1 = m_1g = 2\times10^3g$

$\Rightarrow f_1 = 0.4 \times 2000g = 7848N$

$f_2 = \mu_2 \times R_2$ but $R_2 = m_2g = 1 \times 10^3g$

$\Rightarrow f_2 = 0.2 \times 1000g = 1962N$

For the car,

Net force $= T - f_2$

$\Rightarrow m_1a = F - T - f_2$

$\therefore 2000 \times 1 = F - 2962 - 7848$

$\Rightarrow F = 12810N$

F is the tractive or driving force

(ii)

Power $= \frac{Workdone}{time\ taken}$ But Workdone = Force × distance

$\Rightarrow Power = \frac{F\times d}{t} = F \times \frac{d}{t}$, But $\frac{d}{t}$ = velocity

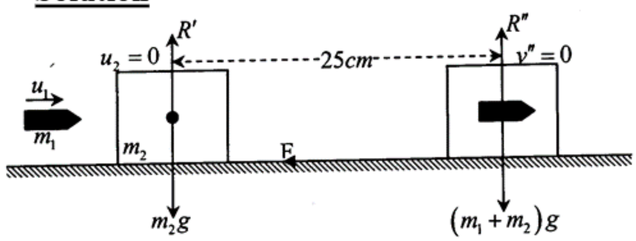
$\therefore Power\ developed = Fv$

But $v = u + at \Rightarrow v = 0 + 1 \times 2.5 = 2.5ms^{-1}$

$\therefore Power\ developed = 12810 \times 2.5 = 32025Watts$

7. A 5g bullet is fired horizontally into a 2.995kg wooden block resting on a horizontal surface is 0.2. The bullet remains embedded in the block which is observed to slide 25cm along the surface before coming to rest. Find the velocity of the bullet just before collision.

Solution



$m_1 = 5g = 0.005kg$, $m_2 = 2.995kg$

Let v' be the velocity of the block and bullet just after collision

Using the conservation of momentum; $m_1u_1 + m_2u_2 = (m_1 + m_2)v'$

$\therefore 0.005u_1 = 2.995 \times 0 = (0.005 + 2.995)v'$

$\Rightarrow 0.005u_1 = 3v'$

$u_1 = 600v' \ldots\ldots\ldots(i)$

From Newton's third equation of motion, $v^2 = u^2 + 2as$

Considering the composite body just after collision,

$u = v', v = v'' = 0, s = 25cm = 0.25m, a = ?$

From newton's second law, F = ma but $F = \mu R'' \Rightarrow \mu R'' = -ma$

Note that we use the reaction of the composite body (bullet and block)

The negative sign is due to the fact that the force, F is a retarding force, and therefore causes a deceleration, as opposed to an acceleration.

$$\therefore a = \frac{-\mu R''}{m} = \frac{-0.2 \times (0.005+2.995)g}{(0.005+2.995)} = -0.2 \times 9.81 = -1.962\text{ms}^{-2}$$

Therefore, $0 = (v')^2 + 2 \times (-1.962)0.25$

$v' = 0.99\text{ms}^{-1}$

But from equation (i)

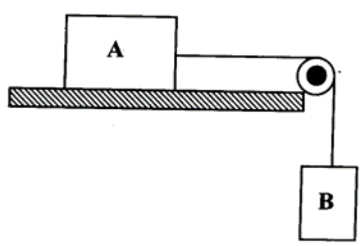
$u_1 = 600v'$

Substituting for v', $u_1 = 600 \times 0.99 = 594\text{ms}^{-1}$

Therefore, velocity of composite body after collision = 594ms^{-1}

Trial questions

1. A bullet of mass 10g fired horizontally with a speed of 200ms^{-1} into a 4kg wooden block at rest on a horizontal surface. The bullet remains embedded in the block which is observed to move a distance of 20m before stopping. Calculate the coefficient of friction between the block and the surface. [Ans : 0.063]
2. In the figure below, masses A and B are 6kg and 1kg respectively, and the coefficient of friction between body A and the table is 0.2.



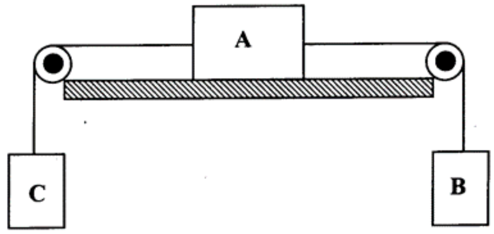
If the system is released from rest, find the frictional force experienced by A and state whether motion will occur or not.

[Ans: 9.81N, no]

3. If for the figure in question 2 , A and B are respectively 1kg and 500g and that the coefficient of friction between A and the table is $\frac{1}{3}$, and the system released from rest with A 3m away from the pulley, and B 2.5m above the floor, find the :
   (i) Initial acceleration of the system
   (ii) Speed with which B hits the floor
   (iii) Speed with which A hits the pulley

   [Ans: 1.09ms^{-2} , $2\frac{1}{3}$ ms^{-1} , 1.48ms^{-1}]

4. In the figure below, A, B and C are masses of 5kg, 2kg and 3kg respectively. When the system is released from rest, body B descends with an acceleration of $0.28ms^{-2}$.



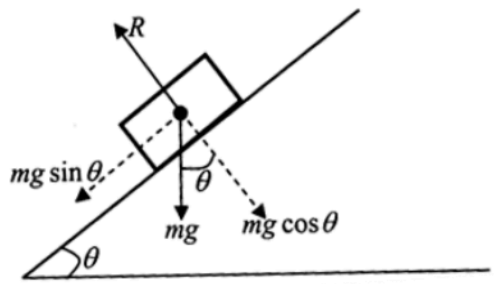
Find the coefficient of friction between body A and the table. [Ans: $\frac{1}{7}$]

5. Two particles A and B of mass 2kg and 4kg respectively are connected by a light inextensible string which passes over a smooth pulley attached to the edge of a rough horizontal table. The particle A is held at rest on the table 4m from the pulley with the string taut, and B hangs vertically below the table and A is $\frac{3}{5}$ .A is released. Find the magnitude of the tension in the string as well as the acceleration of each particle. After B has fallen a distance 2m, it hits the ground and does not rebound. Find the distance from the pulley at which A comes to rest.

[Ans: $\frac{32}{15}g$ N , $\frac{7}{15}g\ ms^{-2}$, $\frac{4}{9}m$]

MOTION ALONG INCLINED PLANES

1. Smooth surface



Consider a body of mass m placed or resting on a plane which is inclined at an angle θ to the horizontal. The plane exerts a normal reaction R, which by newton's third law should be equal to the component of the weight perpendicular to the plane.

$$\Rightarrow R = mg\cos\theta$$

Also, revolving parallel to the plane shows that the component of weight down the plane is $mg\sin\theta$.

- If the body is moving down the plane;

  Net downward force = $mg\sin\theta$

  $$\Rightarrow ma = mg\sin\theta$$

  $$\therefore a = \text{g}\sin\theta$$

- If the body is moving up the plane

  Net upward force = $-mg\sin\theta$

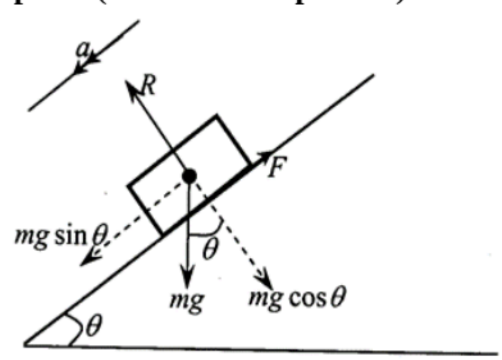
  $$\Rightarrow ma = -mg\sin\theta$$

  $$\therefore a = -\text{g}\sin\theta$$

However, it should be noted that the motion up the plane cannot take place unless there is an upward force, or the body is given an initial velocity. In either case, the equations should be changed to suit the number of forces acting on the body parallel to the plane.

2. Rough surface

- **Motion down the plane (friction acts upwards)**



$$R = mg\cos\theta$$

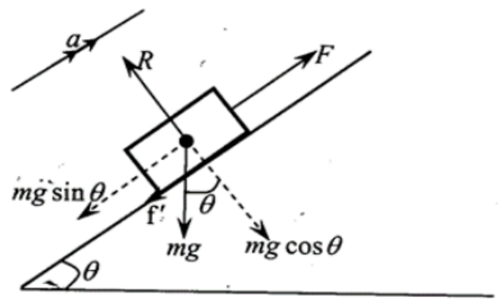
Net downward force = $mg\sin\theta - F$

But F $= \mu R = \mu\, mg\cos\theta$

$\Rightarrow ma = mg\sin\theta - \mu\, mg\cos\theta$

$\therefore a = (\sin\theta - \mu\cos\theta)g$

❖ **Motion up the plane (friction downwards)**



Assuming that an upward force F acts parallel to the plane (upwards), such that it causes the body to move upwards, and assuming that the frictional force is f'.

$R = mg\cos\theta$

Net upward force $= F - mg\sin\theta - f'$

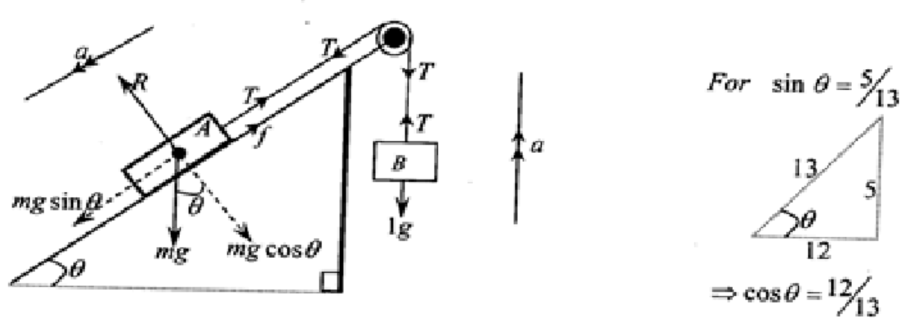
But $f' = \mu R = \mu\, mg\cos\theta$

$\Rightarrow ma = F - mg\sin\theta - \mu\, mg\cos\theta$

Examples

1. A body A of mass 13kg lies on a rough plane inclined an angle θ to the horizontal, where $\sin\theta = \frac{5}{13}$. The coefficient of friction between the plane and the body is 0.1. From A, a light inextensible string passes up the line of greatest slope and over a smooth pulley, to a body B of mass 1kg. If the system released from rest, find the;
   (i) Acceleration of the system
   (ii) Tension in the string
   (iii) Distance moved by A after 3s

   Solution



(i) Consider body A:

$f = \mu R, \mu = 0.1,\ R = mg\cos\theta = 13 \times \frac{12}{13} \times 9.81$

$\Rightarrow f = 0.1 \times 12 \times 9.81 = 11.772N$

Net force = $13\,g\sin\theta - f - T$

$\Rightarrow ma = 13 \times 9.81 \times \frac{5}{13} - 11.772 - T$

$\therefore 13a + T = 37.278$ ....(i)

Consider body B:

Net force = T – 1g

$\Rightarrow 1 \times a = T - g$

$a - T = -9.81$ ....(ii)

Adding two equations (i) + (ii);

$$\begin{array}{rl} & 13a + T = 37.278 \\ + & a - T = -9.81 \\ \hline & 14a + 0 = 27.268 \end{array}$$

$a = 1.692\text{ms}^{-2}$

(ii) From equation (ii) , $T = a + 9.81$

$\Rightarrow T = 1.692 + 9.81 = 11.772N$

(iii) From newton's second equation of motion

$S = ut + \frac{1}{2}at^2$

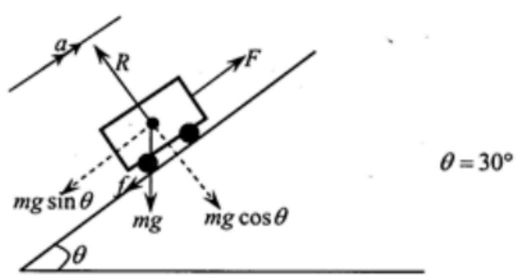
$u = 0, a = 1.962, t = 3s$

$S = 0 \times 3 + \frac{1}{2} \times 1.962 \times 3^2$

$S = 8.83m$

2. An old car of mass 2.0×10^3 kg and tractive pull of 4.0×10^3N climbs a track which is inclined at an angle of 30^0 to the horizontal. The velocity of the car at the bottom of the incline is 30ms^{-1} and the coefficient of sliding friction is $\frac{1}{4}$. Calculate the;
   (i) Distance traveled along the incline before the car makes a halt
   (ii) Time taken

Solution



$m = 2.0 \times 103,\ \mu = 0.25, u = 30, v = 0, F = 4.0\ \times 103$

$f = \mu R = 0.25 \times 2000 \times 9.81 \times \cos 30$

$\Rightarrow f = 4247.85N$

Net force = $F - f - mg\sin\theta$

$\Rightarrow 2000a = 4000 - 4247.85 - 2000 \times 9.81 \times \sin 30$

$\therefore a = \frac{-10057.85}{2000} = -5.03\text{ ms}^{-2}$

From $v^2 = u^2 + 2as$

$0 = 30^2 + 2 \times (-5.03) \times s$

$\therefore s = 89.46m$

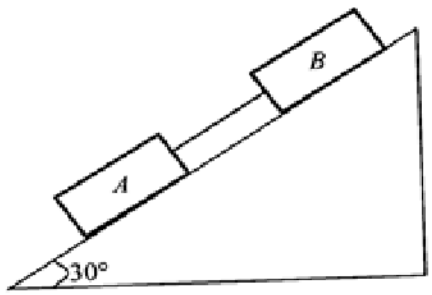
(ii)

From $v = u + at$,

$0 = 30 + (-5.03)t$

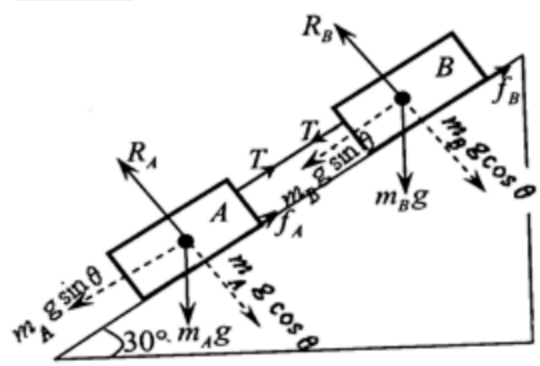
$\therefore t = 5.96s$

3. In the figure shown, blocks A and B have masses 8kg and 16kg respectively, and are connected by a light rigid rod. The blocks slide down the plane, and the coefficient of kinetic friction between block A and the plane is 0.25, while that between block B and the plane is 0.5.



Calculate the acceleration of each block, and the tension in the rod.

Solution



Consider block A:

$\mu_A = 0.25, R_A = m_A g \cos 30 = 8g\frac{\sqrt{3}}{2}$

$f_A = \mu_A R_A = 0.25 \times 4g\sqrt{3}$

$= 16.99N$

Net force $= 8g \sin\theta - T - f_A$

$\Rightarrow 8a = 8 \times 9.81 \times \sin 30 - T - 16.99$

$8a = 22.25 - T \ldots\ldots\ldots (i)$

Consider block B:

$\mu_B = 0.5, R_B = m_B g \cos 30 = 16g\frac{\sqrt{3}}{2}$

$f_B = \mu_B R_B = 0.5 \times 8g\sqrt{3}$

$= 67.97N$

Net force $= T - f_B + 16g \cos\theta$

$\Rightarrow 16a = T - 67.97 + 78.48$

$16a = T - 10.51 \ldots\ldots\ldots (ii)$

Adding the two equations gives;

$24a = \ 32.76$

$\therefore a = 1.36\text{ms}^{-2}$

Substituting for a in equation (i) gives;

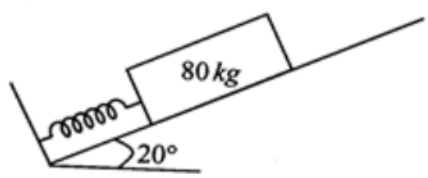
$8 \times 1.36 + T = 22.25$

$\therefore T = 11.4N$

Trial questions

1. A body of mass 2kg is held at rest on a rough sloping roof at a distance of 2m from the edge of the roof. The roof is inclined at an angle $\alpha = \sin^{-1}\frac{3}{5}$ to the horizontal and the edge of the roof is 14m above the horizontal ground floor. The body is then given an initial velocity of 1ms^{-1} down the line of greatest slope. It falls to the ground vertically and rebounds to a height of 5m. If the coefficient between the roof and the body is 0.25, find the speed of the body;
   (i) At the edge of the roof
   (ii) Just after the first collision with the ground floor
   (iii) If the collision lasts 0.02s, find the average force which the body floor exerts on the body. [Ans: 4.086ms^{-1} , 9.9ms^{-1}, 2665N]
2. A mass of 4kg lies on a rough plane which is inclined at 30^0 to the horizontal. A light inelastic string has one end attached to its mass, passes up the line of greatest slope, over a smooth fixed at the top of the plane and carries a freely hanging mass of 1kg at its other end. The system is released from rest with coefficient of friction being $\frac{1}{5}$. Calculate the:
   (i) Acceleration of the system,
   (ii) Tension in the string,
   (iii) Kinetic energy of the 1kg after 3s

   [Ans: 0.603ms^{-2} , 9.207N , 6.54J]
3. In the figure shown, the 80g block is placed on an inclined plane, and the block rests against a spring. The system is released from rest.



   If the coefficient of static friction between the block and the incline is 0.25,
   (i) Determine the maximum and minimum values of the initial compression force in the spring for which block will not slip upon release,
   (ii) Calculate the magnitude and direction of the frictional force acting on the block if the spring compression force is 200N

   [Ans: (i) F_{max} = 453N, F_{min} = 84N, (ii) $f = 68.4N$ up the incline]
4. A body of mass 100g rests on a rough horizontal surface and has a light string, inclined at 20^0 above the horizontal, attached to it. When the tension in the string is $5\times 10^{-1}N$, the

body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the surface. What would the tension in the string have to be for the body to accelerate along the surface at 1.5ms^{-2} ?

[Ans: 0.48 , 0.66N]

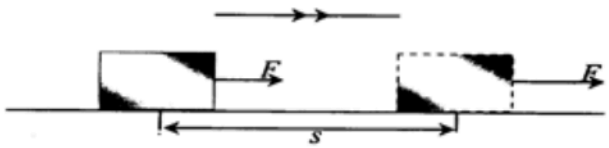
5. A force F acting parallel to and up a rough plane of inclination θ, is just sufficient to prevent a body of mass m from sliding down the plane. A force 4F acting parallel to and up the same rough plane causes the mass m to be on the point of moving up the plane. If μ is the coefficient of friction between the mass and the plane, show that $5\mu = 3\tan\theta$
6. A particle of mass 250g is released from rest at the top of a rough plane which is inclined at $\sin^{-1}\frac{3}{5}$ to the horizontal. The coefficient of friction between the particle and the plane is $\frac{11}{18}$ and the plane is of length 2.5m. Find whether the particle will slide down the plane, and if it does, find its speed on reaching the bottom. [Ans: yes, $2\frac{1}{3}$ms^{-1}]
7. A body slides down a rough plane inclined at 30^0 to the horizontal, the coefficient of kinetic friction between the body and the plane is 0.4, find the velocity after it has travelled 6m along the plane. [Ans: 4.25ms^{-1}]

CHAPTER 5: WORK ENERGY AND POWER

WORK

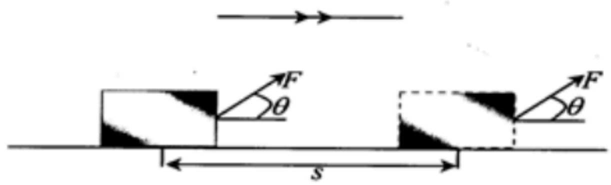
When a body is moved by action of a constant force, the work done is given by the product of the force in the direction of motion and the distance moved.

Consider the figure below in which a constant force Facts on a body such that the body moves through distance s



The force is in the direction of motion, and therefore; Work done $= F \times s$

However, if the force acts at an angle θ to the horizontal as shown below,



The component of force in the direction of motion is Fcos θ

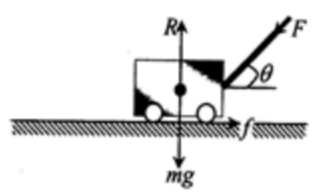
Therefore the work done to move the body through a distance s is, $(\mathrm{F}\cos\theta) \times s$

$$\therefore Work\ done = \mathrm{Fs}\cos\theta$$

Note:

- The SI unit of work done is Nm or Joules
- Work done is a scalar quantity

We should therefore be in position to explain why it is easier to pull a lawn mower than to push it. Consider a gentleman pushing a lawn mower as shown below



Forces acting on lawn mower

The applied force F has a vertical (down ward component) $Fsin\ \theta$. This component increases the weight of the lawn mower, and from Newton's third law, the normal reaction increases. Since the frictional force f is directly proportional to the normal reaction, i.e $f \propto R$, then the frictional force increases, hence making the pushing difficult. However, in pulling, the vertical component of the applied force is upwards, and hence reduces the resultant weight. The normal reaction reduces, and hence the frictional force, which makes the pulling easier.

POWER

This is the rate of doing work or the rate of energy

$$\therefore Power = \frac{workdone}{time\ taken} = \frac{F \times s}{t} = F \times \frac{s}{t}$$

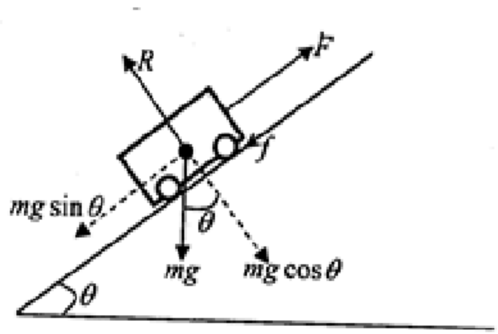
But $\frac{s}{t} = velocity \Rightarrow Power = Fv$

The SI unit of power is Js^{-1} or Watts

Examples

1. A train of mass 500 tonnes has a maximum speed of $90kmh^{-1}$ while moving up an inline of $\sin^{-1}\frac{1}{50}$ against frictional resistance of 100,000N. Find the maximum power of the engine.

   Solution



$\theta = \sin^{-1}\frac{1}{50}$, $\Rightarrow \sin\theta = \frac{1}{50}, f = 100000, m = 500000$

$V_{max} = 90kmh^{-1} = \frac{90 \times 1000}{3600} = 25ms^{-1}$

When velocity is maximum, acceleration is zero

$$\therefore Net\ force = F - f - mg\sin\theta$$
$$\Rightarrow 500000 \times 0 = F - f - mg\sin\theta$$
$$\therefore F = f + mg\sin\theta$$
$$= 100000 + 500000 \times 9.81 \times \frac{1}{50}$$
$$= 198100N$$

But $Power = Fv$

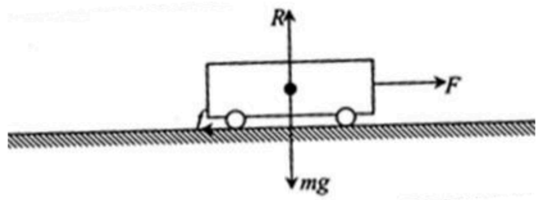
$\Rightarrow$ When speed is maximum, power is also maximum, such that, $P_{max} = Fv_{max}$

$P_{max} = 198100 \times 25$

$= 4952500$ Watts

2. A car of mass 750kg resting on a level road is uniformly accelerated for 10seconds, until the speed is $18kmh^{-1}$. If the resistance to motion is 5gN, calculate the power of the car 10 seconds after starting the motion.

Solution



$m = 750, v = 18kmh^{-1} = \frac{18\times1000}{3600}$ms^{-1}, $= 5$ms^{-1}, $u = 0, t = 10$

From $v = u + at$

$5 = 0 + 10a \Rightarrow a = 0.5$ms^{-2}

Net force = $F - f$

$\Rightarrow ma = F - f$

$750 \times 0.5 = F - 5 \times 9.81$

$\therefore F = 424.05N$

But P = Fv

$\Rightarrow P = 424.05 \times 5 = 2120.25\ Watts$

Trial questions

1. A body of mass 10kg is pulled a distance of 20m across a horizontal surface against resistance totaling 40N. If the body moves with a uniform velocity, find the work done against the resistances. [Ans: 800J]
2. A rough surface is inclined at an angle ϕ to the horizontal. A body of mass m is pulled at a uniform speed x up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the plane is μ. If the only resistances to motion are those due to gravity and friction, show that the work done on the body is; $mgx(\sin\phi + \mu\cos\phi)$ where g is the acceleration due to gravity
3. A car of mass 800kg, working at a constant rate of 15000W ascends a hill of $\sin^{-1}\frac{1}{98}$ against a constant resistance to motion of 420N. Find ;
   (i) The acceleration of the car up the hill when traveling with a speed of 10ms^{-1}
   (ii) The maximum speed of the car up hill,
   (iii) The acceleration of the car down the same hill at the instant when its speed is 20ms^{-1}, resistance being constant.
   [Ans: 1.25ms^{-2} , 30ms^{-1} , 0.5125ms^{-2}]

ENERGY

This is the ability of doing work, or it's the measure of the capacity to do work. Therefore, if a force does work on a body, it changes the energy of the body. Mechanical energy is divided into two i.e kinetic energy and potential energy.

Kinetic energy

This is the energy of a body by virtue of its motion

Therefore, when a force does work on a body so as to increase its speed, then the work done is measure of the increase in the kinetic energy of the body.

Suppose a constant horizontal force F acts on a body of mass m which is initially at rest on a smooth horizontal surface, such that after moving through a distance s along the surface, the body has a velocity v.

$$Work\ done = F \times s \qquad but\ F = ma$$

$$\Rightarrow W.d = mas \ldots..(i)$$

From Newton's third equation of motion, $v^2 = u^2 + 2as$

Since u = 0, $v^2 = 2as$

$$\Rightarrow a = \frac{v^2}{2s}$$

Substituting for a in equation (i)

$$W.d = m \times \frac{v^2}{2s} \times s = \frac{1}{2}mv^2$$

The quantity $\frac{1}{2}mv^2$ is defined as the kinetic energy of a body of mass m moving with a velocity of v. i.e

Kinetic = $\frac{1}{2}mv^2$.

It can therefore be noted that a body be noted that a body at rest has zero kinetic energy. However, if the velocity of the body had changed from u to v, then from $v^2 = u^2 + 2as$

$$\Rightarrow a = \frac{v^2 - u^2}{2s}$$

And substituting for a in equation (i) above gives;

$$W.d = m \times \frac{v^2 - u^2}{2s} \times s = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

The quantity $\left(\frac{1}{2}mv^2 - \frac{1}{2}mu^2\right)$ is equal to the change in kinetic energy of a body of mass m from an instant when its velocity is u to an instant when its velocity is v.

This therefore means that the work done is equal to the change in kinetic energy. This is the work-energy theorem.

The work-energy theorem states that the work done by a resultant force on a body is equal to the change in kinetic energy of the body.

Examples

1. A cycle of mass 200kg traveling at 144kmh^{-1} on a horizontal road is brought to rest in a distance of 80m by action of brakes, and frictional forces. Find the
   (i) average stopping force
   (ii) time taken to stop the cycle

Solution

$m = 200kg, u = 144kmh^{-1} = \frac{144 \times 1000}{3600}$ = 40ms^{-1} , s = 80m, v = 0

Initial kinetic energy = $\frac{1}{2}mu^2 = \frac{1}{2} \times 200 \times 40^2 = 160000$J

Final kinetic energy = $\frac{1}{2}mv^2 = \frac{1}{2} \times 200 \times 0 = 0$

$\therefore change\ in\ kinetic\ energy = 160000 - 0 = 160000$

Work done = change in kinetic energy

$$\Rightarrow F \times s = 160000$$

$\therefore F = \frac{160000}{80} = 2000N$

(ii)

From F = ma, $a = -\frac{F}{m}$ the negative sign means that the car is decelerating or retarding

$\therefore\ a = -\frac{2000}{200} = -10\text{ms}^{-2}$

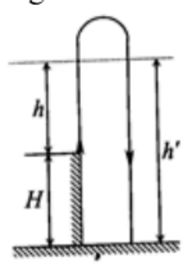
From $v = u + at, 0 = 40 + (-10)t \Rightarrow 10t = 40 \qquad \therefore t = 4s$

Note that one could get the same results by using $v^2 = u^2 + 2as$ to find the value of the acceleration.

2. An object is projected with a velocity u at a height H vertically upwards,
   (i) Explain how its kinetic energy varies with height above the ground
   (ii) Sketch a graph to show the relationship in (i) above

Solution

Let the body be projected vertically upwards with velocity u, and v be its velocity at a height h. In the diagram, h is the height above the point of projection as it ascends.

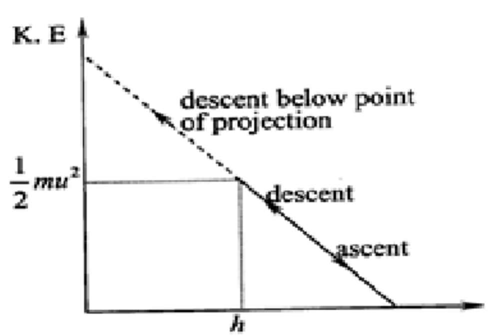


$v^2 = u^2 - 2gh$

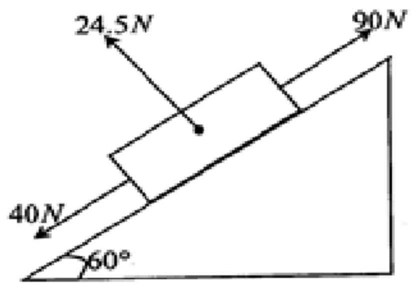
Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m(u^2 - 2gh) = \frac{1}{2}mu^2 - mgh$

This is the kinetic energy of the body as it ascends. It can therefore be seen from the expression of kinetic energy that;

- K.e decreases linearly with height h
- K.e becomes zero when the body is instantaneously at rest when it reaches maximum height
- As the particle begins to descend, its kinetic energy begins to increase linearly distance descended.



3. In the figure shown, three forces 90N, 40N and 24.5N act on a block placed on a smooth plane inclined at an angle of 60^0 to the horizontal.

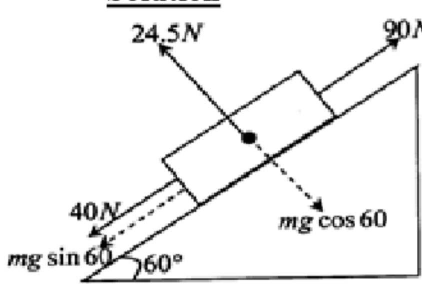


Calculate the:

(i) Acceleration of the block

(ii) Gain in kinetic energy 5seconds after moving from rest.

Solution



(i) If the block is not to move off the plane, then $24.5 = mg\cos 60$

$$\Rightarrow 24.5 = m \times 9.81 \times \cos 60$$

$$\therefore m = 5\text{kg}$$

Considering the upward motion along the plane;

$$Net\ force = 90 - (40 + mg\sin 60)$$

$$\therefore ma = 90 - 40 - mg\sin 60$$

$$\Rightarrow 5a = 50 - 5 \times 9.81 \times \frac{\sqrt{3}}{2}$$

$$a = 1.5\text{ms}^{-2}$$

(ii) From Newton's second equation of motion

$$v = u + at \qquad But\ u = 0\ ,\quad a = 1.5\ \ and\ t = 5$$

$$\therefore v = 0 + 1.5 \times 5 = 7.5\text{ms}^{-1}$$

Gain in kinetic energy = Final kinetic energy − Initial kinetic energy

$$= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

But u = 0, $\Rightarrow \frac{1}{2}mu^2 = 0$

$$\therefore Gain\ in\ kinetic\ energy = \frac{1}{2}mv^2 = \frac{1}{2} \times 5 \times (7.5)^2 = 140.6\text{Joules}$$

Trial questions

1. A body of mass 6kg, initially moving with a speed of 12ms^{-1}, experiences a constant retarding force of 10N for 3 seconds. Find the kinetic energy of the body at the end of this time. [Ans: 147J]
2. A cricketer throws a ball of mass 0.40kg directly upwards with a velocity 40ms^{-1} and catches (receives) it again 8.0s later. Draw labeled sketch graphs to show:

(i) The velocity
(ii) Kinetic energy
(iii) Height of the ball against time over the stated 8.0s. Show the numerical values of the given quantities.

Potential energy

This is the energy possessed by a body by virtue of its position above the ground. Therefore if a body of mass m kg is raised vertically through a distance h metres, then,

$Workdone\ against\ gravity = mgh$, where g is the acceleration due to gravity

Similarly, if a body is lowered vertically, its potential energy decreases.

It should therefore be noted that the work done against gravity is a measure of potential energy.

$\Rightarrow Potential\ energy = mgh$

This is called gravitational potential energy

However, a stretched or compressed spring also has potential energy, called elastic potential energy.

Elastic potential energy $= \frac{1}{2}ke^2$ where; e is the extension or compression of the spring, k is the force constant of the spring.

The constant of the spring k is obtained from Hooke's law i.e $F \propto e$

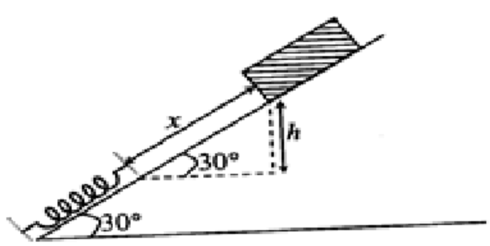
$\Rightarrow F = ke$

$\therefore k = \frac{F}{e}$ The SI unit of k is Nm^{-1}

Examples

1. A block of mass 0.2kg is released from rest at the top of a smooth plane inclined at 30^0 to the horizontal. The block compresses a spring placed at the bottom of the plane by 10cm before it momentarily comes to rest. If the force constant of the spring is $20Nm^{-1}$, determine the distance the block has travelled down the incline before it comes to rest and its speed just before it reaches the spring.

Solution



Mass = 0.2kg, u = 0, e = 10cm = 0.1m , k = $20Nm^{-1}$

Elastic potential energy $= \frac{1}{2}ke^2 = \frac{1}{2} \times 20 \times 0.1^2 = 0.1$Joules

Gravitational potential energy = mgh but $h = x \sin 30$

$= mg \times \frac{1}{2}x = 0.2 \times 9.81 \times 0.5x = 0.981x$

But from the conservation of energy,

Gravitational potential energy = Elastic potential energy

$\Rightarrow 0.981x = 0.1$

$\therefore x = 0.1m$

Kinetic energy = potential energy

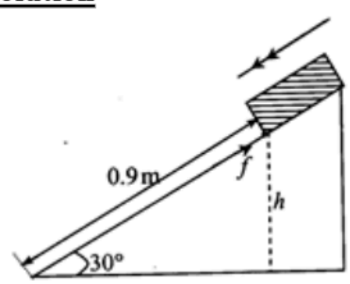
$\Rightarrow \frac{1}{2}mv^2 = 0.1$

$\therefore \frac{1}{2} \times 0.2 \times v^2 = 0.1$

$v = 1ms^{-1}$

2. A 12kg block is released from released from rest from a rough plane inclined at30^0 to the horizontal from a point 0.9m from the base. The coefficient of kinetic friction between the block and the inclined plane is 0.25.
   (i) With what speed will the block reach the bottom of the incline?
   (ii) If the block is projected up the incline with a speed of $20ms^{-1}$, how far up the incline will the block travel?

Solution



$h = 0.9\sin 30$

$\Rightarrow h = 0.45m$

(i) Potential energy lost by the block = Kinetic energy gained + work done against friction in moving down the incline

Potential energy = mgh, Kinetic energy = $\frac{1}{2}mv^2, work\ done = f \times d$

Where: v is the velocity of the bottom of the incline, f the frictional force, and d the distance travelled by the block along the plane.

$\Rightarrow mgh = \frac{1}{2}mv^2 + f \times d$ ....(i)

$f = \mu R \ \ where\ R = m\,g\cos\theta$

$\Rightarrow f = \mu\, m\,g\cos\theta = 0.25 \times 12 \times 9.81 \times \cos 30 = 25.49N$

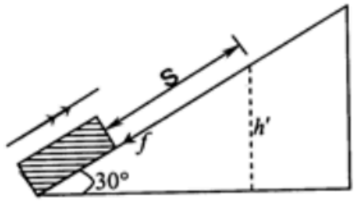
Thus from equation (i);

$12 \times 9.81 \times 0.45 = \frac{1}{2} \times 12 \times v^2 + 0.9 \times 25.49$

$52.974 = 6v^2 + 22.941$

$\therefore v = 2.24ms^{-1}$

(ii)



$$h' = s\sin 30 \Rightarrow \ h = \frac{s}{2}$$

$u = 20, v = 0, f = 25.49N$

Kinetic energy lost = Potential energy gained + work done against friction

$$\Rightarrow \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgh' + f \times s$$

$$\frac{1}{2} \times 12 \times 20^2 = 12 \times 9.81 \times \frac{s}{2} + 25.49 \times s$$

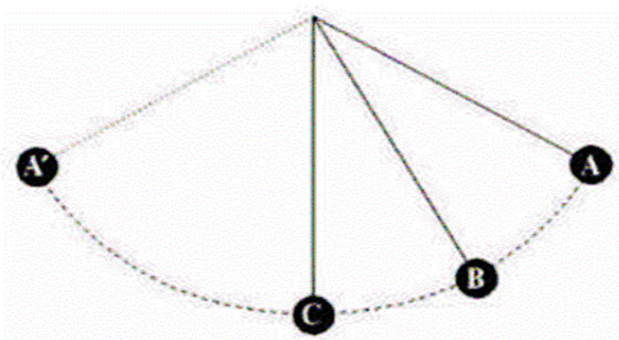
$$2400 = 58.86s + 25.49s$$

$$\Rightarrow s = 28.45m$$

3. (a) Explain the energy changes which occur when a pendulum is set into motion.
   (b) A simple pendulum of length 1m has a bob of mass 100g. It is displaced from its mean position A to a position B so the string makes an angle of 45^0 with the vertical. Calculate the;
   (i) Maximum potential energy of the bob
   (ii) Velocity of the bob when the string makes an angle of 30^0 with the vertical [Neglect air resistance]

Solution

(a)



Before the bob is released at point A, it possesses potential energy. When the bob is released, the potential energy starts reducing as it goes to equilibrium but as potential energy reduces, Kinetic energy is gained as it goes through point B and increases until it reaches the equilibrium position (Point C). At equilibrium, all the potential energy is lost and therefore the bob contains only kinetic energy which makes it rise again as it starts losing the kinetic energy and gains potential energy at point A'.

In summary; At A: Potential energy is maximum, Kinetic energy = 0

At B: Potential energy and Kinetic energy

At C: Kinetic energy, Potential energy = 0

At A': *Potential energy, Kinetic energy = 0*

(b) (i)

At B;

$h = l - l\cos 45 = l(1 - \cos 45)$ $but\ l = 1m$

$\Rightarrow h = 1 - \cos 45 = 0.293m$

$Maximum\ potential\ energy = mgh$ but m =100g = 0.1kg

= 0.1 × 9.81 × 0.293 = 0.2874J

(ii) The potential energy is lost as it is converted to kinetic energy, thus when the sting makes an angle of 30^0, the bob possesses both kinetic energy and potential energy

When the string makes an angle of 30^0,

$h = l - l\cos 30 = l(1 - \cos 30)$ $but\ l = 1m$

$\Rightarrow h = 1 - \cos 30 = 0.134m$

Potential energy = $mgh = 0.1 \times 9.81 \times 0.134 = 0.1315$J

Kinetic energy = $\frac{1}{2}mv^2 = \frac{1}{2} \times 0.1 \times v^2 = 0.05v^2$

From the principle of conservation of mechanical energy;

$0.1315 + 0.05v^2 = 0.2874$

$0.05v^2 = 0.1559 \Rightarrow v^2 = 3.118$

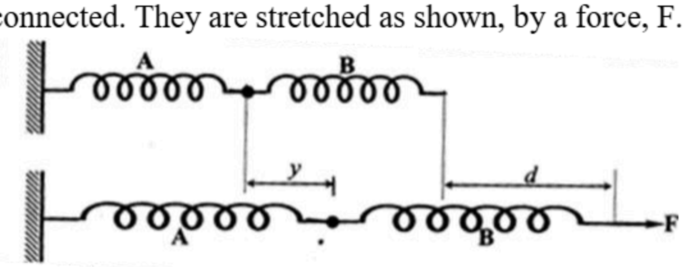
$\therefore v = 1.766$ms^{-1}

Trial questions

1. A bullet of mass 10g is fired at short range into a block of mass 990kg resting on a smooth horizontal surface and attached to a spring of force constant 100Nm^{-1}. The bullet remains embedded in the block while the spring is compressed by a distance of 5.0cm. Find the :
   (i) Elastic potential energy
   (ii) Speed of the bullet just before collision with the block

   [Ans (i) 1.25 ×10^{-1} Joules, v = 50 ms^{-1}]
2. A constant force pulls a body of mass 500g along a straight, smooth, horizontal surface. The body passes through points P and Q is at distance of 3m from P, find the magnitude of the constant force acting on the body.

   [Ans: $1\frac{1}{3}N$]
3. In the figure below, two springs A and B of force constants k_1 and k_2 respectively are connected. They are stretched as shown, by a force, F.



If the net force at the at the contact is zero, show that $y = \frac{dk_2}{k_1+k_2}$

Pump raising and ejecting water

Consider a pump which is used to raise water from a source (well or tank) and then eject it at a given speed. The total work done is the sum if the potential energy in raising the water, and the kinetic energy given to the water. The work done per second gives the rate (power) at which the pump is working.

work done per second = P.E given to the water per second + K.E given to water per second

Example

A pump draws 3.6m^3 of water of density 1000kgm^{-3} from a well 5m below the ground in every minute, and issues it at ground level through a pipe of cross sectional area 40 cm^2. Find:

(i) The speed with which water leaves the pipe,

(ii) The rate at which the pump is working

(iii) If the pump is only 80% efficient, find the rate at which it must work.

Solution

(i)

Volume = 3.6m^3 and time = 60s $\Rightarrow volume\ per\ second = \frac{3.6}{60}$

But volume per second = Area × $velocity = \frac{40}{10000}v$

$\therefore \frac{40}{10000}v = \frac{3.6}{60}$

$\Rightarrow v = 15ms^{-1}$

(ii)

Mass per second = $(volume\ per\ second) \times (density) = \frac{3.6}{60} \times 1000 = 60kgs^{-1}$

Kinetic energy per second = $\frac{1}{2} \times (mass\ per\ second) \times v^2 = \frac{1}{2} \times 60 \times 15^2 = 6750Js^{-1}$

Potential energy per second = $(mass\ per\ second) \times g \times h = 60 \times 9.81 \times 5 = 2943Js^{-1}$

Power = 6750 + 2943 = 9693 W

(iii)

Power output = 9693 W, Power input = ? , Efficiency = 80%

Efficiency = $\frac{Power\ output}{power\ input} \times 100\%$

$\therefore 80\% = \frac{9693}{P_{in}} \times 100\%$

$\Rightarrow P_{in} = 12116.25\ W$

Trial question

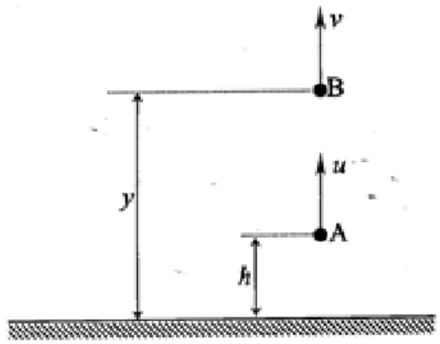
A water pump working at a constant rate of 900W draws 0.3m^3 of water from a deep well and issues it through a nozzle situated 10m above the level from which the water was drawn after every minute. If the pump is 75% efficient, find the velocity with the water is ejected and the cross sectional area of the nozzle. [Ans: 8.6ms^{-1}, 5.81cm^2]

Principle of conservation of mechanical energy

It states that in any mechanical system, the total energy remains constant provided that there are no dissipative forces.

Dissipative forces are forces where work done against such a force cannot be recovered. Examples of such forces are; air resistance, friction, viscous drag e.t.c

Consider a body of mass m projected vertically upwards with a speed u from a point A, a height h above the horizontal ground. Assume that the body has a velocity v at a point B, a height y from the same ground. Also assume that throughout its upward motion, air resistance is negligible.



At A:

Kinetic energy, k.e = $\frac{1}{2}mu^2$ and potential energy, p.e = mgh

But Mechanical energy, m.e = k.e + p.e

$\Rightarrow m.e_A = \frac{1}{2}mu^2 + mgh$

At B:

Kinetic energy, k.e = $\frac{1}{2}mv^2$ and potential energy, p.e = mgy

But Mechanical energy, m.e = k.e + p.e

$\Rightarrow m.e_B = \frac{1}{2}mv^2 + mgy$

From the principle of conservation of mechanical energy, $m.e_A = m.e_B$

$\therefore \frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mgy$

Proof of the principal of conservation of mechanical energy

Consider the situation above;

At point B; $s = y - h$ and $a = -g$

From $v^2 = u^2 + 2as$, $v^2 = u^2 - 2g(y - h)$

Multiplying through by $\frac{1}{2}m$ gives: $\frac{1}{2}mv^2 = \frac{1}{2}m[u^2 - 2g(y - h)]$

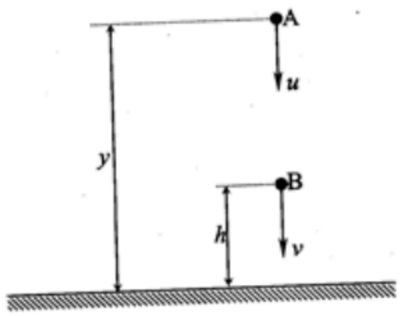
$\therefore \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mgy + mgh$

$\Rightarrow \frac{1}{2}mv^2 + mgy = \frac{1}{2}mu^2 + mgh$

$\therefore m.e_A = m.e_B$

Alternatively;

Consider a body released from a point A such that it falls to the ground,



Since the body is just released, u = 0

$\therefore\ k.e_A = \frac{1}{2}mu^2 = 0 \ \ and \ \ p.e_A = mgy$

$m.e_A = k.e_A + \ \ p.e_A = mgy$

At B;

$s = y - h \quad and \ \ a = g$

From $v^2 = u^2 + 2as \quad \Rightarrow \ \ v^2 = 2as = 2g(y - h)$

Kinetic energy at B, $k.e_B = \frac{1}{2}mv^2 = \frac{1}{2}m[2g(y - h)]$

$= mgy - mgh$

Potential energy at B, $p.e_B = mgh$

$m.e_B = k.e_B + \ \ p.e_B = (mgy - mgh) + \ mgh$

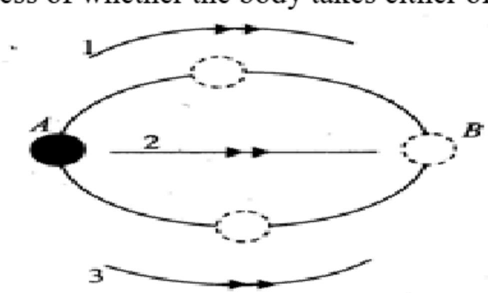
Principle of conservation of energy

It states that energy can neither be created nor destroyed but may be changed from one form to another. Therefore the total energy of a system is always constant.

Conservative forces

These are forces for which;

- The work done to move a body from one point to another is independent of the path taken and only depends on the initial and final positions of the body. In the figure, if a conservative force acts on a body at A, the work done in moving the body from A to B is the same regardless of whether the body takes either of the paths 1, 2 or 3



- The work done in moving a body round a closed path is zero. Therefore in the figure, if a conservative force acts on a body at A such that the body is moved from A to any point and then back to A, the work done by such a force is zero.
- Mechanical energy of a body onto which a conservative force acts is always a conserved

Examples of conservative forces include; gravitational force, electrostatic force and elastic force.

Non-conservative forces are forces for which the work done against such a force in moving a body from one point to another depends on the path taken, the work done to move a body round a closed path is not zero, and mechanical energy of a body onto which such a force acts is not conserved.

Examples of non-conservative forces include; air resistance, friction, viscous drag etc.

CHAPTER 6: STATICS

Moment

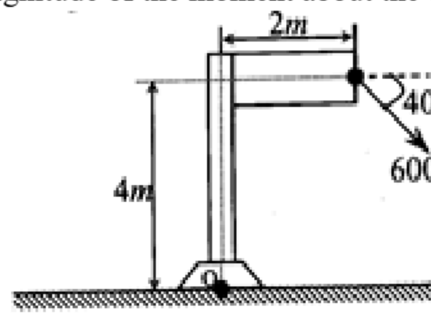
Moment of a force is the product of the force and the perpendicular distance of action from the pivot.

Moment = $Force\ \times Perpendicular\ distance$

Therefore, a force cannot have a moment about a point on its line of action. The SI unit of moment is Nm and not Joules.

Example

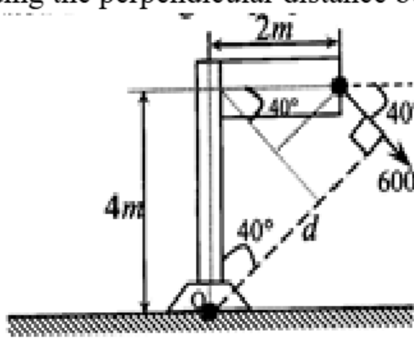
Calculate the magnitude of the moment about the base point O of the 600N force.



Solution

There are two methods that can be used to find the required moment

Method 1: Finding the perpendicular distance between the force and the pivot



d is the perpendicular distance

From the figure; $d = 4\cos 40 + 2\sin 40$

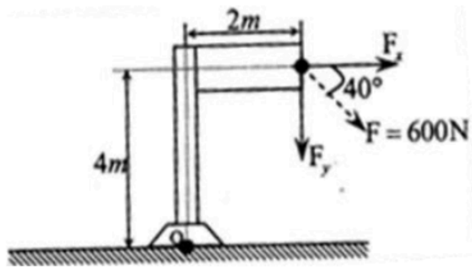
$= 4.35$m

From Moment = Fd,

$Moment = 600 \times 4.36 = 2610$N

Note: This method requires that the perpendicular distance is interpreted and found correctly

Method 2: Finding the components of the force, the perpendicular distances from which to the pivot can easily be found.



$F_x = 600\cos 40 = 459.63N$ and $F_y = 600\sin 40 = 385.67N$

Perpendicular distance of F_x from O is 4m

Perpendicular distance of F_y from O is 2m

Moment due to F_x = $459.63 \times 4 = 1838.52Nm$

Moment due to F_y = $385.67 \times 2 = 771.34Nm$

Total moment at O = Moment due to F_x + Moment due to F_y

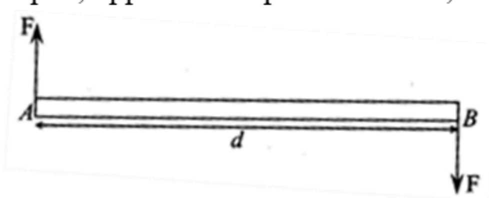
= 1838.52 + 771.34

= 2610Nm as obtained before

It can therefore be noted from the above example that the moment of a force about any point is equal to the sum of the moments of the force about the same point.

COUPLE

A couple is pair of equal, opposite and parallel forces, whose lines of action do not coincide.

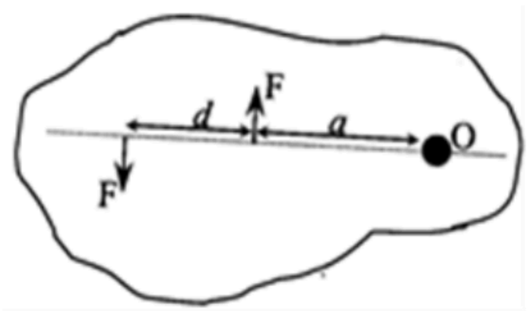


Consider two equal and opposite forces of magnitude F acting at a distance d apart. The two forces constitute a couple. These two forces cannot be combined into a single force since their sum in every direction is zero. Their effect is entirely to produce a tendency of rotation. Moment of a couple is the product of one of the forces and the perpendicular distance between the forces. The moment of a couple is called torque

$$\therefore Torque = F \times \overline{AB}$$

Also consider the two forces to be acting in a plane of a card board as shown in the figure below: If the two forces act such that they cause a tendency of rotation in the anticlockwise direction about point O, then total moment is the summation of the two moments

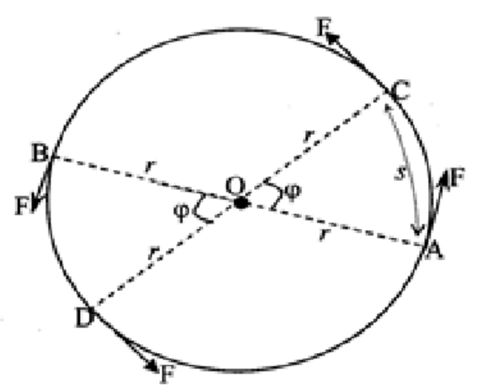
$$\Rightarrow M = -F \times a + F(d + a) = Fd$$



Since the moment of a couple contains no reference to the dimensions of a which locates the forces with respect to the pivot O, it follows that the moment of a couple is independent of the pivot position for as long as it is on the line of the perpendicular distance between the two forces. Therefore, it following points should specifically be noted about couples and moments. They can be referred to as **properties of a couple**;

- A couple tends to produce rotation
- The moment of a couple is the same about any point in a plane of the lines of action of the forces that constitute the couple.
- A couple cannot be replaced by a single force, but can be replaced by any other couple of the same moment,
- The resultant of two or more couples is a couple whose moment is the sum of the moments of the individual couples.

Work done by a couple



Consider two equal and opposite forces F acting along the tangent of a wheel of radius r. If the couple causes the wheel to rotate through an angle φ from AB to CD as shown above,

Distance moved through by each force, s $= \frac{\varphi}{360} \times 2\pi r$

But $360^0 = 2\pi$ radians

Therefore, if $\varphi$is in radians, then $\frac{\varphi}{2\pi} \times 2\pi r = \varphi r$

Since Work done = Force × distance, then;

Work done by each force = F φr

But there are two forces acting on the wheel ⇒ $Total\ workdone = 2F\varphi r = (2Fr) \times \varphi$

But torque = $Force \times Perpendicular\ distance\ between\ the\ forces$

$= F \times 2r = 2Fr$

$\therefore$ Total work done by a couple = $(Torque) \times \varphi$

NB: φ should be in radians

The reader should note that an understanding of the knowledge about forces, couples and moments is essential for the study of equilibrium in the proceeding section. This is due to the fact that many of the difficulties students find in applying equilibrium principles can be traced to errors arising from failure to observe and apply correctly procedures of these concepts

EQUILLIBRIUM

Statics as part of physics primarily deals with the description of the conditions of the force which are both necessary and sufficient to maintain the state of equilibrium. This part of equilibrium therefore takes the central part of statics, and should therefore be mastered.

A particular body is said to be in equilibrium if a number of forces act on it but it does not move or rotate.

Conditions of equilibrium

1. The resultant of all forces that act on a body should be zero. The components of forces in both x and y directions should be zero. This implies that the acceleration of the body should be zero, and so the body should be in translational equilibrium.
2. The sum of clockwise moments about any point must be equal to the sum of anticlockwise moments about the same point. This is the principle of moments. It can also be stated as: The algebraic sum of moments of the forces about any point should be zero. This implies that the body should be in rotational equilibrium.

Types of equilibrium

1. **Stable equilibrium:**
   When a body in stable equilibrium is tilted, its centre of gravity is raised above the original position. On release, it returns to its original position without toppling.

2. **Unstable equilibrium:**
   This is when a body is slightly displaced, the centre of gravity is lowered hence it cannot return to its original position

3. **Neutral equilibrium:**
   When a body is slightly displaced, the centre of gravity and potential energy are unchanged and constant.

How to improve stability

- Increasing the base area
- Lowering the centre of gravity

Centre of mass:

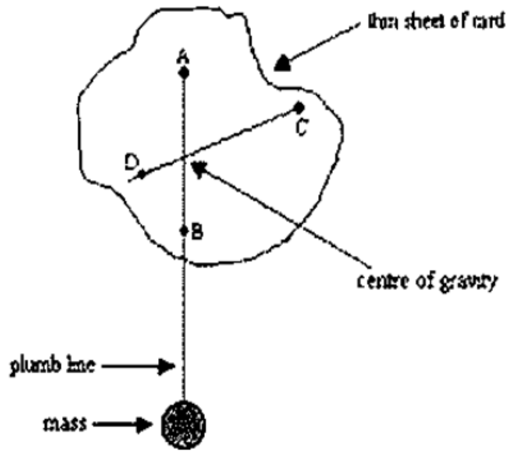
This is the point at which an applied force produces acceleration but no rotation.

Centre of gravity:

Centre of gravity of a body is the point of application of the resultant force of attraction on the body due to the earth.

Experiment to determine the centre of gravity of an irregular lamina

Three holes are made at well-spaced intervals around the edge of the lamina. The lamina is suspended on a clamped pin passing through one of the holes and the plumb line is hung from the pin so that it is in front of the lamina.



The plumb line position is marked along the laminar by two pencil marks that are later joined to make a line.

The procedure is repeated with one of the other holes and the plumb line position marked as before.

The position of intersection of the two lines is the centre of gravity.

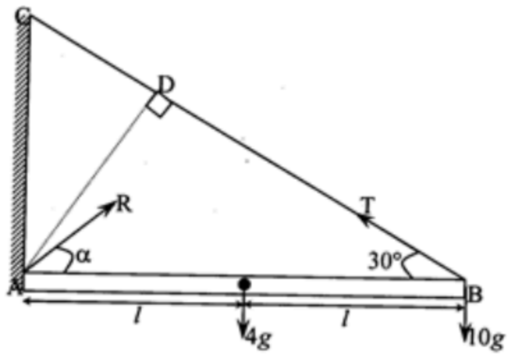
Examples

1. A uniform beam AB of mass 4kg is hinged to a wall at end A and held horizontally by a wire joining B to a point C which is on the wall vertically above A. If a stone of mass 10kg is hinged at B, and given that $\angle ABC$ is 30^0 , find the force:
   (i) In the wire connecting B to C
   (ii) Exerted by the beam on the hinge

Solution

(i) Taking moments about point A,

T is anticlockwise, 4g and 10g are clockwise, and R does not have a moment about A



Let $\overline{AB} = 2l$

$\Rightarrow \overline{AD} = 2l\sin 30°$

$= 2l \times \frac{1}{2} = l$

$$sum\ of\ clockwise\ moments = sum\ of\ anti\ clockwise\ moments$$

$$4g \times l + 10g \times 2l = T \times l$$

$$24gl = Tl$$

$$\therefore T = 24g = 24 \times 9.81 = 235.44\ N$$

(ii)

Resolving;

(i) Resolving,

Forces	*Horizontally*(→)	*vertically* (↑)
4g	0	- 4g
10g	0	-10g
T	-235.44 cos 30	235.44sin 30
R	R cos α	R sin α
	$\sum F_x = R\cos\alpha - 203.9$	$\sum F_y = R\sin\alpha - 19.62$

Since the body is in equilibrium, then $\sum F_x = 0$ *and* $\sum F_y = 0$

$$\therefore R\cos\alpha - 203.9 = 0 \quad \text{and} \quad R\sin\alpha - 19.62 = 0$$

$$R\cos\alpha = 203.9 \ldots\ldots (i) \qquad R\sin\alpha = 19.62 \ldots\ldots\ldots (ii)$$

Dividing eqn (ii) by eqn (i) gives; $\frac{R\sin\alpha}{R\cos\alpha} = \frac{19.62}{203.9}$

$$\therefore \tan\alpha = 0.09622 \Rightarrow \alpha = 5.496^0$$

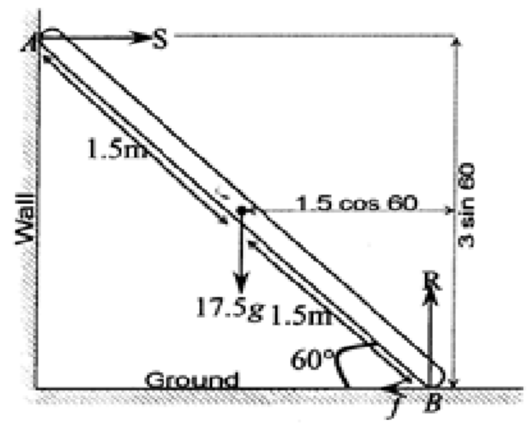
From equation (i) , $R = \frac{203.9}{\cos 5.496} = 204.8N$

Therefore, the force at the hinge is 204.8N at an angle of $5.496^0 to\ AB$.

2. A uniform ladder 3m long and of mass 17.5kg, rests with its upper end A against a smooth vertical wall, and the lower end B on a rough ground. What must be the coefficient of friction between the ground and the ladder be for the ladder to be inclined at 60^0 to the horizontal without slipping? What is the reaction at B?

Solution

Let R and S be the normal reactions from the ground on to the ladder, and from the wall onto the ladder respectively. Since the ladder is in equilibrium, then the sum of forces in any particular direction should be zero.



Vertically (↑): $R = 17.5\text{g}$

$\Rightarrow R = 17.5 \times 9.81 = 171.68N$

Horizontally (→) : $S = f$

Taking moments about point B:

$S \times 3\sin 60 = 17.5\text{g} \times 1.5\cos 60$

$S = \frac{17.5 \times 9.81 \times 1.5 \times \cos 60}{3 \times \sin 60} = 49.56N$

But $S = f \quad \Rightarrow f = 49.56N$

Since $f = \mu R$; $\mu = \frac{f}{R} = \frac{49.56}{171.68} = 0.29$

Total reaction at B = $\sqrt{f^2 + R^2}$

= $\sqrt{49.56^2 + 171.68^2}$

= 178.7N

If the reaction is at an angle θ to the horizontal, then $\tan\theta = \frac{R}{f} = \frac{171.68}{49.56}$

$\theta = 73.9^0$

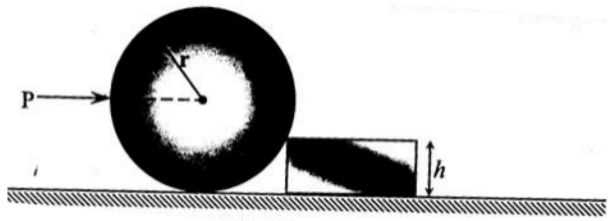
The following points about solving problems of equilibrium should be noted:

- It is important to draw a clear diagram that shows all the forces acting on the body, or ladder like in the above case
- Much as such things as angles should give a clear view of the angles they represent, there is no need of using a protractor to draw the diagram
- A smooth surface can only exert a force at right angles to itself – the normal reaction
- There is no point in resolving in more than two directions
- It is always an advantage to resolve perpendicular to an unknown force
- It is usually an advantage to take moments about points where unknown forces are acting.

Trial questions

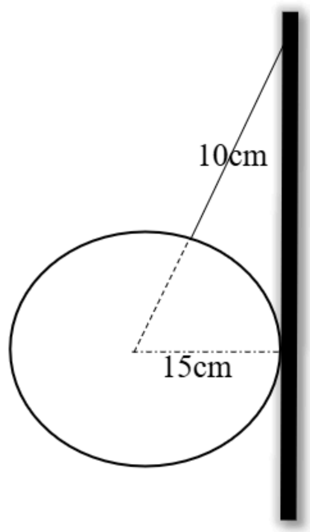
1. A beam of mass 20kg and length 2.4m is hinged at A which is a point on a vertical wall. The beam is maintained in a horizontal position by means of a chain attached to a point in a wall 1.5m vertically above A. If the beam carries a load of 10kg at a point 1.8m from A, calculate the:
   (i) Tension in the chain
   (ii) Magnitude and direction of the reaction at A
   [Ans: T = 323.73N and R = 301.17N]
2. A uniform rod AB of length $4x$ and weight 30N is smoothly hinged at its upper end, A . The rod is held at 30^0 to the horizontal by a string which is at 90^0 to the rod, and attached to it at C where AC = $3x$. Find the ;
   (i) Tension in the string
   (ii) Vertical component of the reaction at A
   (iii) Horizontal component of the reaction at A
   [Ans: (i) T = 17.4N, (ii) 15.0N upwards (iii) 8.7N away from B]
3. A sphere of weight 40N and radius 30cm rests against a smooth vertical wall. The sphere is supported in this position by a string of length 20cm attached to a point on the wall. Find the; (i) tension in the sting
   (ii) Reaction due to the wall
   Hint: Make use of the fact that there are three forces actin on the sphere
   [Ans: T = 50N, R = 30N at 90^0 to the wall]
4. A uniform beam of mass m and length $2l$ has its lower end A resting on a rough horizontal ground and is kept in equilibrium, at an angle of 45^0 to the vertical, by a rope attached to end B. If the rope makes an angle of 60^0 with BA, and given that the tension in the rope is T, show that, T = $\frac{mg}{\sqrt{6}}$, hence, or otherwise, find the tension in a rope which is used to hold a uniform beam AB of mass 100kg and length $2l$ with its lower end A resting on a rough horizontal ground, and making an angle of 20^0 with the horizontal, such that the rope makes an angle of 40^0 with BA. [Ans: T = 716N]
5. A rigid rod PQ has length 2m. A body of mass 12kg hangs from P and another body of mass 8kg hangs from Q. The system is suspended from a point A of the rod, where A is x m from Q, and is in equilibrium with the rod horizontal. Find the value of x
   [Ans: $x = 1.2\ m$]
6. A uniform ladder of mass 40kg and length 5m, rests with its upper end against a smooth vertical wall and with its lower end at 3m from the wall on a rough ground. Find the magnitude and the direction of the force exerted at the bottom of the ladder.
   [Ans: 418.7N at 69.4^0 to the horizontal]
7. A mass of 5kg is suspended from end A of a uniform beam of mass 1.0kg and length 1.0m. The end B of the beam is hinged in a wall. The beam is kept horizontal by a rope attached to A and to a point C, in the wall at a height 0.75m above B.
   (i) Draw a sketch diagram to show the forces acting on the beam
   (ii) Calculate the tension in the rope
   (iii) What is the force exerted by the hinge on the beam.
   [Ans: T = 89.N, R =72.0 at 3.9^0 above the horizontal]

8. In the figure below, if a horizontal force P is required to begin rolling the uniform cylinder of mass m over the obstruction of height h,



Show that $P = \frac{mg\sqrt{2rh-h^2}}{r-h}$

9. If an oil drum of diameter 70cm and mass 80kg rests on a stone like in the diagram in question 8, find the least horizontal force applied through the centre of the drum which will cause the drum to roll over the stone of height 10cm. [Ans: 471.6N]
10. A sphere of weight 20N and radius 15cm rests against a smooth vertical wall. The string is supported in this position by a string of length 10cm attached to the point on the sphere and to a point on the wall as shown below.

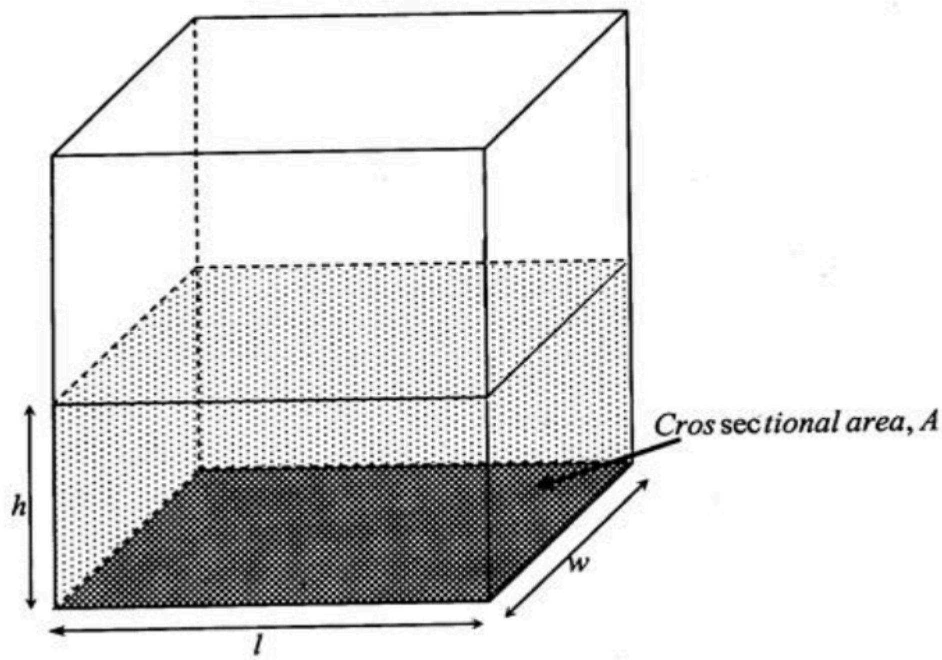


(i) Copy the diagram and show all the forces that are acting on the sphere
(ii) Calculate the reaction on the sphere due to the wall
(iii) Find the tension in the string [Ans: R =15N, T =25N]

PRESSURE IN FLUIDS

Pressure is the force exerted normally on a surface of area $1m^2$. The SI unit of pressure is Nm^{-2} or Pa. Pressure in fluids is a scalar quantity since it is transmitted equally in all directions, and so there is no single direction along which it acts.

Derivation of hρg



Consider a column of liquid of depth h and density h in a container of cross sectional area A.

Volume of liquid = $l \times w \times h = Ah$

Mass of liquid = $V\rho$ = $Ah\rho$

Weight of liquid = mg = $Ah\rho g$

This weight is the force exerted on the container by the liquid.

But pressure = $\frac{Force}{Area} = \frac{Ah\rho g}{A}$

$= h\rho g$

Therefore, the pressure in liquids is independent of the cross – sectional area, and shape of the vessel in which it's poured. The pressure acts in all directions, and it depends only on the depth, h, the density, ρ and on the atmospheric pressure.

Density: Is the ratio of mass to volume. ⇒ Density = $\frac{m}{v}$. The SI unit of density is kgm^{-3}

Other units used include gcm^{-3}

$1\ gcm^{-3} = 1000kgm^{-3}$

Experiment to determine the density of an irregular solid which floats in water

A thread is tied to the irregular solid and its weight in air W_1 determined. A sinker such as a stone is attached to the irregular solid, and the weight W_2 of the two in air is determined. The sinker and the solid are completely immersed in water of known density and their weight W_3 is determined by a spring balance. The sinker is detached from the solid and the weight W_4 of the sinker only when in water is determined.

Weight of irregular solid in water = $W_3 - W_4$

$\Rightarrow$ Relative density of irregular solid = $\frac{W_1}{W_1-(W_3-W_4)}$

$\therefore$ From Density = Relative density × Density of water,

The density of the irregular solid can be determined

Relative density (R.D)

$$R.D = \frac{Density\ of\ a\ substance}{Density\ of\ water} = \frac{mass\ of\ substance}{mass\ of\ an\ equal\ volume\ of\ water}$$

$$= \frac{weight\ of\ a\ substance}{weight\ of\ an\ equal\ volume\ of\ water}$$

Any of the above formulas can be used to find the R.D of both liquids and solids.

To find the R.D of solids only

$$R.D = \frac{mass\ of\ substance\ in\ air}{Apparent\ loss\ of\ mass\ of\ the\ substance\ when\ in\ water}$$

$$= \frac{weight\ of\ substance\ in\ air}{Apparent\ loss\ of\ weight\ of\ the\ substance\ while\ in\ water\ (Upthrust)}$$

To find the R.D of liquids only

$$R.D = \frac{apparent\ loss\ in\ mass\ of\ a\ substance\ when\ in\ the\ liquid}{apparent\ loss\ in\ mass\ of\ the\ substance\ when\ in\ the\ liquid}$$

$$= \frac{Apparent\ loss\ in\ weight\ of\ a\ substance\ when\ in\ the\ liquid}{Apparent\ loss\ in\ weight\ of\ the\ substance\ when\ in\ the\ liquid}$$

Example

A block of mass 0.1 kg is suspended from a spring balance. When the block is immersed in water of density 1000kgm^{-3}, the spring balance reads 0.63N. When the block is immersed in a liquid of unknown density, the spring balance reads 0.70N. Find the;

i) Density of the mass,
ii) Density of the liquid.

Solution

i) Mass of solid in air = 0.1kg

$\Longrightarrow$ Weight of solid in air = 0.1×9.81 = 0.981N

Weight of solid in water = 0.63kg

Apparent loss of weight of the solid in water = 0.981 – 0.63 = 0.351N

$$R.D = \frac{weight\ of\ solid\ in\ air}{Apparent\ loss\ of\ weight\ of\ solid\ when\ in\ water}$$

$$= \frac{0.981}{0.351} = 2.795$$

But $R.D = \frac{Density\ of\ solid}{Density\ of\ water}$ $\Rightarrow 2.795 = \frac{\rho_s}{1000}$

Density of solid = 2.795 × 1000 = 2795kgm^{-3}

ii) Apparent loss of weight of solid in liquid = 0.981 – 0.7 = 0.281N

$$R.D\ of\ liquid = \frac{apparent\ loss\ of\ weight\ of\ a\ sold\ in\ the\ liquid}{Apparent\ lopss\ of\ weight\ of\ the\ sold\ while\ in\ water}$$

$$= \frac{0.281}{0.351} = 0.8$$

∴ Density of liquid = 0.8 × 1000 = 800kgm^{-3}

Archimede's principle

It states that when a body is wholly or partially immersed in a fluid, it experiences an upward force (up thrust) which is equal to the weight of the fluid displaced.
It should be noted that up thrust is also called buoyant force, and so the tendency of a fluid to exert an upward force on a body immersed in it is called buoyancy;

Examples

1. A string supports a solid iron object of mass 0.81kg totally immersed in a liquid of density 800kgm^{-3}. Calculate the tension in the string if the density of iron is 8000kgm^{-3}.

Solution

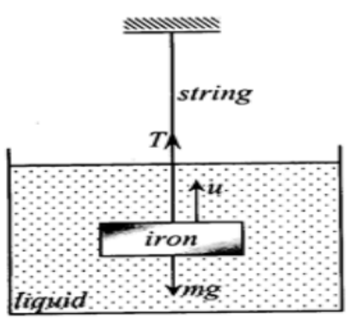
Weight of iron = mg = 0.18 × 9.81 = 1.77N

Volume of iron = $\frac{mass}{density} = \frac{0.18}{8000} = 2.25 \times 10^{-5}$ m^3

∴ Volume of liquid displaced = 2.25×10^{-5} m^3

⇒ Mass of liquid displaced = $(2.25 \times 10^{-5}) \times 0.081 = 0.018$

⇒ Weight of liquid displaced = 0.018 × 9.81 = 0.177N.



But from Archimedes principle, up thrust is equal to the weight of the liquid displaced.

∴ u = 0.177N

But from equilibrium, Mg = T + u
$\Rightarrow$ 1.77 = T + 0.17, T = 1.593N

2. A specimen of an alloy of silver and gold, whose densities are 10.5gcm^{-3} and 18.9 gcm^{-3} respectively, weighs 35.2g in air and 33.13g in water. Find the composition by mass of the alloy, assuming that there has been no volume change in the process of producing the alloy. Assume that the density of water is 1gcm^{-3}

Solution

$\rho_s = 10.5\ gcm^{-3}, \rho_g = 18.9\ gcm^{-3}$

Let the volume of silver be v_s and that of gold be v_g,
Also, let the mass of silver be m_s and that of gold be m_g.
Mass of alloy in air, $m_s = 35.2$g .
Mass of alloy in water, $m_g = 33.13$g

$\Rightarrow m_s + m_g = 35.2$ ……………… (i)

R.D of alloy = $\frac{Mass\ of\ alloy\ in\ air}{Apparent\ loss\ in\ mass\ while\ in\ water} = \frac{35.2}{2.07} = 17$

Density of alloy = R.D × density of water $= 17 \times 1 = 17gcm^{-3}$

Volume of alloy = $\frac{35.2}{17} = 2.07cm^3$

Volume of alloy = $v_s + v_g = \frac{m_s}{\rho_s} + \frac{m_g}{\rho_s} = \frac{m_s}{10.5} + \frac{m_g}{18.9}$

$\therefore$ 2.07 = $\frac{m_s}{10.5} - \frac{m_g}{18.9}$ ……………………… (ii)

But from equation (i), m_g= 35.2 - m$_s$

Substituting for m_g in equation (ii) gives: 2.07 = $\frac{m_s}{10.5} - \frac{35.2 - m_s}{18.9}$

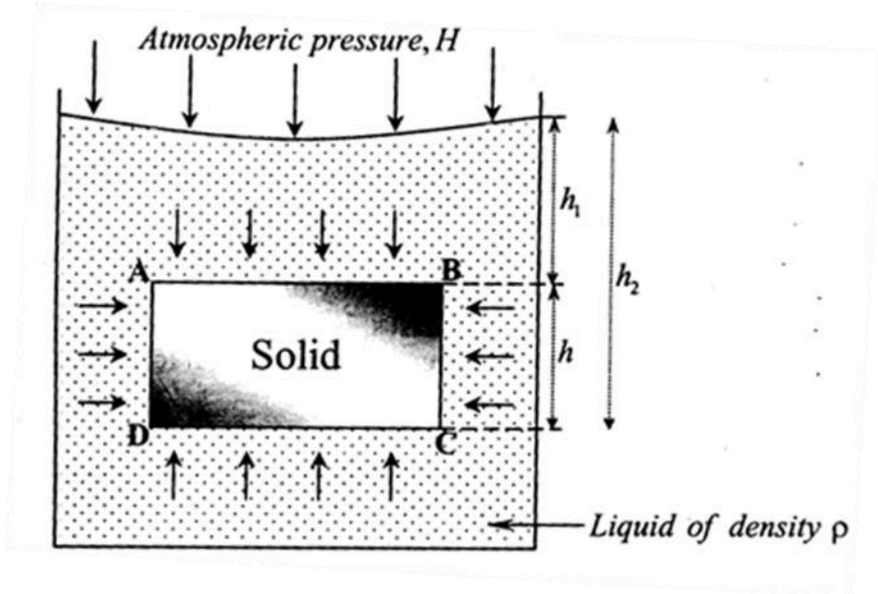
$\therefore$ 410.79 = 369.6 – 10.5m_s + 18.9m_s $\Rightarrow m_s = 4.9$g

Substituting for m_s *in equation (i)gives* : $m_g = 35.2 - 4.9 \Rightarrow m_g = 30.3$g

Trial questions

1. A piece of metal of mass 2.60g and density 8.4g/cm^3 is attached to a block of wax of mass 1.0 g and density 0.92g/cm^3. When the system is placed in a liquid, it floats with wax just submerged. Find the density of the liquid.
[Ans: $\rho = 1.13 \times 10^{-6}$ gcm^{-3}]
2. A cubical block of wood of side 12cm floats at the interface between oil and water with its lower surface 4cm below the interface the heights of the oil and water columns are 10cm each. If the density of oil is 0.8gcm^{-3} and that of water is 1 gcm^{-3}, calculate the ;
i) Mass of the block,
ii) Pressure on the lower surface of the block.

Proof of Archimedes principle



Consider a solid body of cross-sectional area A immersed in a liquid of density ρ.
Pressure on side AB = H + $h_1\,\rho$g ⇒ downward force on AB = A (H +$h_1\,\rho$g)
Pressure on side CD = H + $h_2\,\rho$g ⇒ Upward force on side CD = A (H +$h_2\,\rho$g)

Since $h_2 > h_1$, the upward force is greater than the downward force

⇒ Resultant upward force (up thrust) = A ($H + h_2\,\rho$g) – A (H + $h_1\,\rho$g)

∴ Up thrust = A ρg ($h_2 - h_1$) ……………………. (i)

Also volume of solid = Ah ⇒ Volume of liquid displaced = Ah But h = $h_2 - h_1$

⇒ Volume of liquid displaced = A ($h_2 - h_1$)

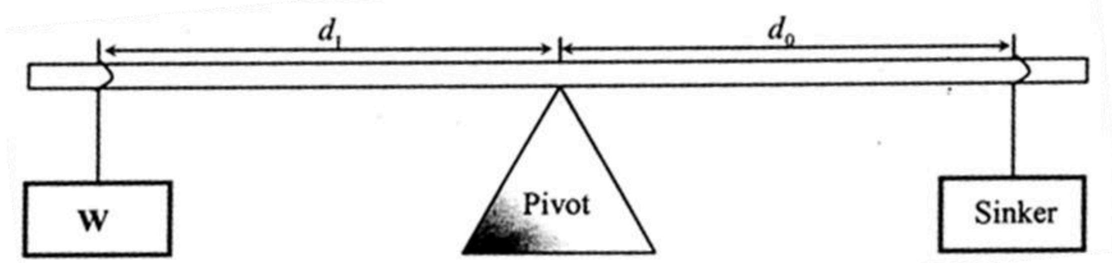
∴ Mass of liquid displaced = A ρ ($h_2 - h_1$)

⇒ $Weight\ of\ liquid\ displaced$ = A ρg ($h_2 - h_1$)……………. (ii)

It can be seen from equations (i) and (ii) that up thrust is equal to the weight of the liquid displaced which is Archimede's principle.

The reader should note that up thrust depends on the density of a body relative to the medium (fluid) in which it is. This explains why a helium filled balloon rises up to a certain height in still air and then stops. Initially, the balloon rises because the up thrust due to air is greater than the weight of the balloon, resulting from the fact that the density of air is greater than that of helium. As the balloon rises up, the density of the air decreases, implying that up thrust also decreases. The balloon stops rising when it's at a height when the up thrust is equal to its weight.

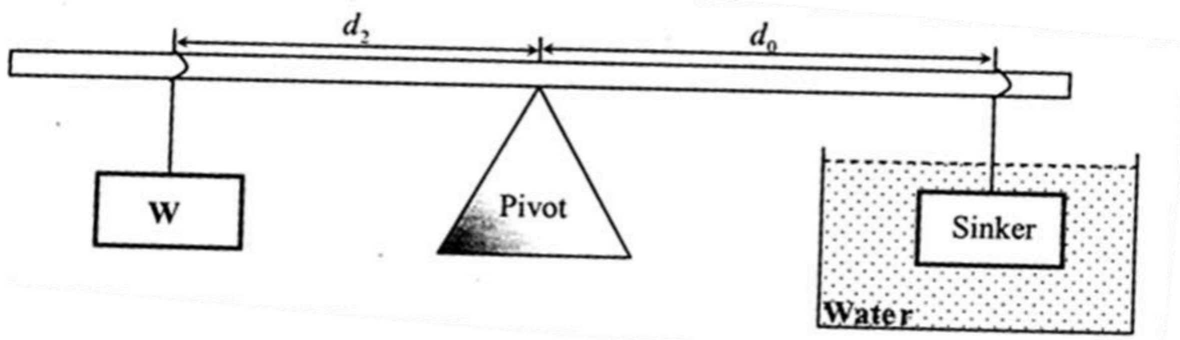
Experiment to determine the relative density to a liquid using Archimedes principle and the principle of moments.



While in air, the sinker (solid) and weight W are attached to the meter rule as shown above. The weight is adjusted until the meter rule balances horizontally. The distances d_1 and d_0 are measured and recorded.

If W_1 is the weight of the sinker in air, then taking moments about the pivot gives:

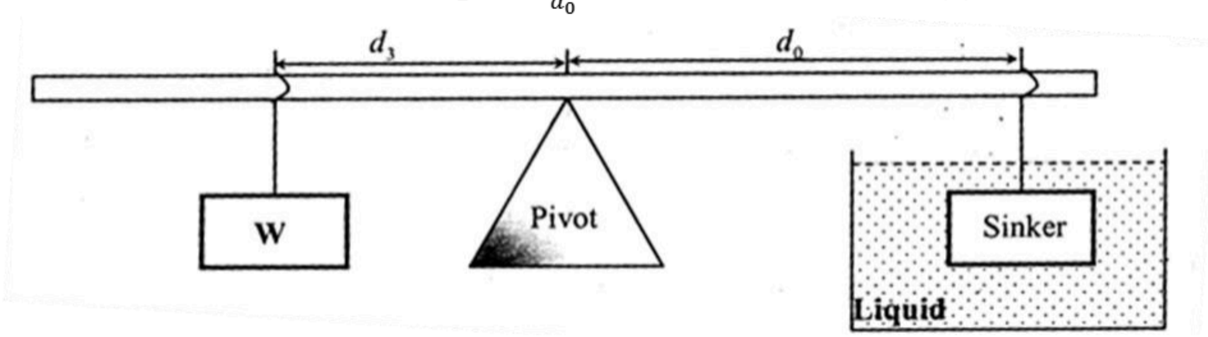
$W_1d_0 = Wd_1 \qquad \Rightarrow W_1 = W\frac{d_1}{d_0}$ .... (i)



The sinker is then immersed in water in a beaker while keeping d_0 constant. The position of the weight W is adjusted until balance is restored. The distance d_2 is measured.

If W_2 is the weight of the sinker in water, then taking moments about the pivot gives:

$W_2d_0 = Wd_2 \quad \Rightarrow W_2 = W\frac{d_2}{d_0}$ .... (ii)



The sinker is then immersed in a liquid in a beaker while keeping d_0 constant. The position of the weight W is adjusted until balance is restored. The distance d_3 is measured. If W_3 is the weight of the sinker in water, then taking moments about the pivot gives:

$W_3d_0 = Wd_3 \qquad \Rightarrow W_3 = W\frac{d_3}{d_0}$ .... (iii)

By definition relative density = $\frac{apparent\ loss\ of\ weight\ of\ the\ sinker\ while\ in\ liqiuid}{apparent\ loss\ of\ weight\ of\ the\ sinker\ while\ in\ water}$

$$(W\frac{d_1}{d_0} - W\frac{d_3}{d_0}) / (W\frac{d_1}{d_0} - W\frac{d_2}{d_0}) = \frac{d_1 - d_3}{d_1 - d_2}$$

Where $d_1 > d_2 > d_3$

It should be noted that one of the advantages of such a method is that relative density can be determined even when the weights are not known.

FLOATATION

A body floats in a fluid if its density is less than that of the fluid in which it is place.

The law of floatation states that a floating body displaces its own weight of the liquid in which it floats.

Proof

For the floating body to be in equilibrium, Up thrust = Weight of floating body

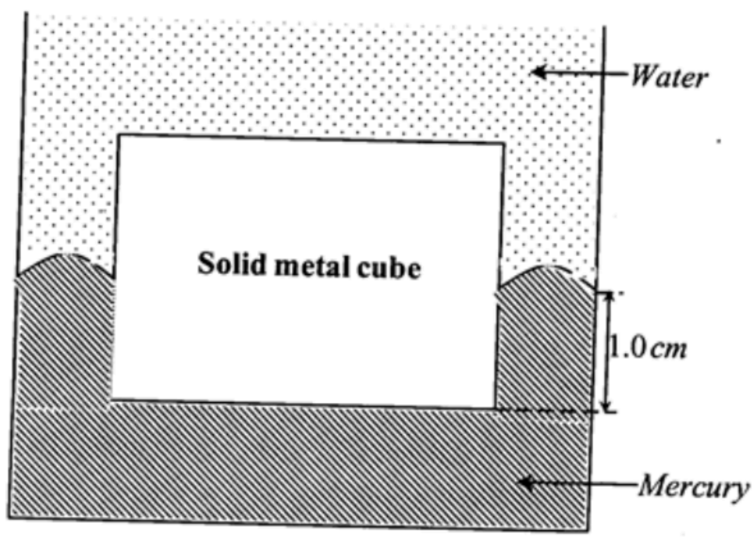
But from Archimedes principle, Up thrust = Weight of fluid displaced

$\therefore$ Weight of floating body = Weight of fluid displaced, which is the law of flotation.

It can therefore be correctly said that: Mass of floating body = Mass of fluid displaced

Examples

1.



A solid metal cube of side 8.0 cm floats vertically at the interface between water and mercury as shown above. The lower surface of the cube is 1 cm below the interface. Given that the density of mercury is 13600kgm^{-3} and that of water is 1000kgm^{-3}, calculate the density of the metal.

Solution

Volume of metal cube $= l \times w \times h = 8 \times 8 \times 8 = 512\text{cm}^3 = 5.12 \times 10^{-4}\ m^3$

Volume of mercury displaced $= 8 \times 8 \times 1 = 64cm^3 = 6.4 \times 10^{-5}\ m^3$

$\Rightarrow$ Mass of mercury displaced = volume $\times$ density $= (6.4 \times 10^{-5}) \times 13600 = 0.87kg$

Volume of water displaced $= 8 \times 8 \times 7 = 448\ cm^3 = 4.48 \times 10^{-4}\ m^3$

$\Rightarrow$ Mass of water displaced = volume × density = $(4.48 \times 10^{-4}\, m^3) \times 1000 = 0.448kg$

$\therefore$ Total mass of liquid displaced = 0.448 + 0.87 = 1.318kg

But from the law of floatation, mass of floating object is equal to mass of liquid displaced

$\therefore$ Mass of the metal cube = 1.318kg

From density = $\frac{mass}{volume}$, density of cube = $\frac{1.318}{5.12 \times 10^{-4}}$

= 2574.2 kgm^{-3}

2. A hydrometer consists of a spherical bulb and cylindrical stem, which has a cross sectional area of 0.6 cm^2. The total volume of the bulb and the stem is 14.3 cm^3. When immersed in water, the hydrometer floats with 7.6 cm of the stem above the water surface. When in alcohol, it floats with 2.0 cm of the stem above the surface. If the density of water is 1 g/cc, calculate the density of alcohol.

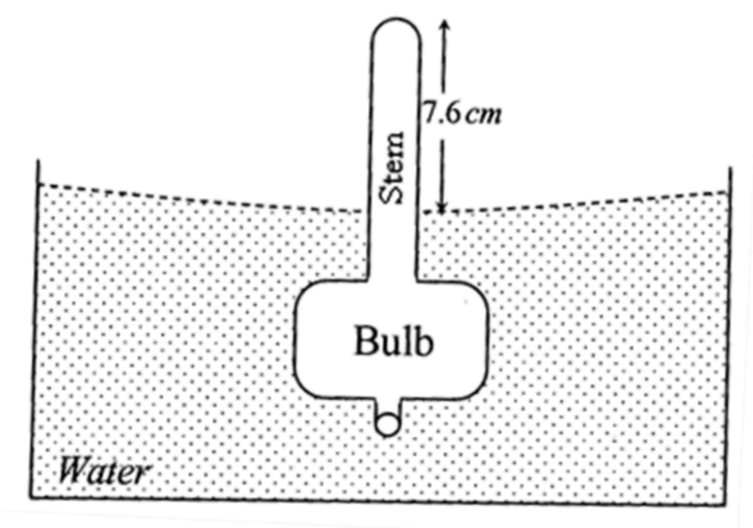
Solution

Volume of hydrometer above water is equal to volume of its stem above the water

= Ah = 0.6 × 7.6 = 4.56 cc

Therefore, volume of hydrometer in water = 14.3 – 4.56 = 9.74 cc.

It therefore follows that volume of water displaced = 9.74 cc.



Mass of water displaced= (volume) × (density) = 9.74 $\times 1 = 9.74$g

From the law of floatation, a floating body displaces its own weight of it liquid in which it floats.

$\Rightarrow$ Mass of hydrometer = 9.74g

When the hydrometer is placed in alcohol:

Volume of alcohol displaced = 14.3 – (Ah) = 14.3 – (0.6 × 2) $= 13.1cc$

Therefore, mass of alcohol displaced = (volume) × (density) = 31.1 × $\rho = (13.1\rho)$g

But, mass of hydrometer should be equal to the mass of alcohol displaced.

$\Rightarrow 9.74 = 13.1\rho \qquad \therefore \rho = 0.744 g/cc$

Therefore, density of alcohol = 0.744g/cc

= 744kgm^{-3}

3. An alloy contains two metals X and Y of densities $3.0 \times 10^3 kgm^{-3}$ and $5.0 \times 10^3\ kgm^{-3}$ respectively. Calculate the density of the alloy if;
   i) The volume of X is twice that of Y,
   ii) The mass of X is twice that of Y.

Solution

(i) Let the volume of Y be v. $\Rightarrow$ Volume of X is 2v.

Also let the mass of X be x, and that of Y be y

From density = $\frac{mass}{volume}$, $3000 = \frac{x}{2v} \Rightarrow x = 6000v \ldots\ldots\ldots\ldots i)\ and$

$$5000 = \frac{y}{v} \Rightarrow v = \frac{y}{5000} \ldots\ldots\ldots\ldots\ldots ii)$$

Substituting for v in equation i) gives:

$$x = 6000\times\left(\frac{y}{5000}\right), \Rightarrow y = \frac{5}{6}x$$

Therefore, total mass of alloy = $x + y = x + \frac{5}{6}x = \left(\frac{11}{6}x\right) kg$

Also, volume of alloy = $2v + v = 3v$

But density = $\frac{mass}{volume} = \frac{(\frac{11}{6}x)}{3v} = \frac{11x}{18v}$, but from equation i), $x = 6000v$

$$\Rightarrow Density = \frac{11\times(6000v)}{18v} = 3.7\times 10^3\ kgm^{-3}$$

(ii) Let the mass of Y be m. $\Rightarrow mass\ of\ X\ is\ 2m.$

Let the volume of X be V_x, and that of Y be V_y.

$$3000 = \frac{2m}{v_x} \quad \Rightarrow v_x = \frac{m}{1500} \ldots\ldots\ldots\ldots\ldots\ldots i)$$

$$5000 = \frac{m}{v_y} \quad \Rightarrow m = 5000v_y \ldots\ldots\ldots\ldots\ldots\ldots ii)$$

Substituting for m in equation i) gives: $v_x = \frac{5000v_y}{1500} \Rightarrow v_y = \frac{3}{10}v_x$

Total volume of alloy = $v_x + v_y = v_x + \frac{3}{10}v_x = (\frac{13}{10}v_x)m^3$

Also, total mass of alloy = 2m + m = (3m) kg

From density = $\frac{mass}{volume}$, Density of alloy = $\frac{3m}{(\frac{13}{10}v_s)} = \frac{30m}{13v_s}$

But from equation i), $v_x = \frac{m}{1500}$.

$\therefore$ Density of alloy = $\frac{30m}{13\times(\frac{m}{1500})} = 3.5\times 10^3\ kgm^{-3}$

4. A simple hydrometer consisting of a loaded glass bulb fixed at the bottom of a glass stem of uniform cross-sectional area sinks in water of density 1.0g/cc, so that a certain mark *x*

on its stem is 4.0 cm below the surface of water. When placed on a liquid of density 0.9g/cc, the hydrometer floats with the mark 6.0 cm below the surface of the liquid. How far below the surface will the mark be if the hydrometer is placed in a liquid of density 1.1g/cc.

Solution

Let the cross sectional area of the stem for the hydrometer be A.
Also, let the weight of the hydrometer in air be W_a.

While in water: Volume of water displaced = 4A,

Mass of water displaced = volume× density = 4A × 1 = 4A

Weight of water displaced = 4Ag

While in the first liquid: Volume of water displaced = 6A,

Mass of water displaced = volume $\times\ density = 6A\ \times 0.9 = 5.4A$

Weight of water displaced = 5.4Ag

While in the second liquid; Let the distance from the surface to the mark *be x*

Volume of water displaced = Ax,

Mass of water displaced = Volume $\times\ density = Ax\ \times 1.1 = 1.1Ax$

Weight of water displaced = $1.1Ax$g

Relative density = $\frac{Apparent\ loss\ of\ weight\ of\ the\ hydrometer\ while\ in\ a\ liquid}{Apparent\ loss\ of\ weight\ of\ the\ hydrometer\ while\ in\ water}$

Considering water and the first liquid; R.D = $\frac{W_a - 5.4Ag}{W_a - 4Ag}$

But relative density is also equal to: R.D = $\frac{Density\ of\ a\ substance}{Density\ of\ water} = \frac{0.9}{1} = 0.9$

$$\therefore 0.9 = \frac{W_a - 5.4Ag}{W_a - 4Ag} \Longrightarrow 0.9\,w_a - 3.6Ag = w_a - 5.4Ag$$

$$\therefore\ w_a = 18Ag \ldots\ldots\ldots\ldots\ldots(i)$$

Considering water and the second liquid; R.D = $\frac{W_a - 1.1Axg}{W_a - 4Ag}$

But Relative density is also equal to: R.D = $\frac{density\ of\ a\ substance}{density\ of\ water} = \frac{1.1}{1} = 1.1$

$$\therefore 1.1 = \frac{W_a - 1.1Axg}{W_a - 4Ag} \Longrightarrow 1.1\,w_a - 4.4Ag = w_a - 1.1Axg$$

$$\therefore\ w_a = 44Ag - 11Axg\ \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Equating the two equations gives: 18Ag = 44Ag – 11Axg $\Longrightarrow\ 26 = 11x$

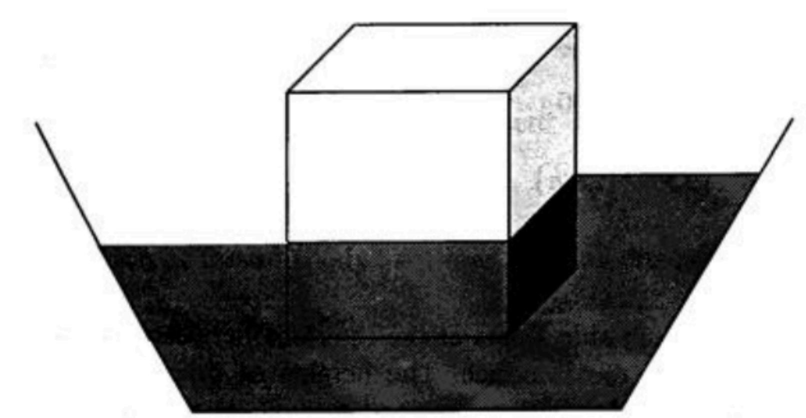
Therefore, x = 2.36 cm

5. A cubical block of steel 12cm on each side is floating on mercury in a vessel. The density of steel is 7800kgm^{-3} and that of mercury is 13600kgm^{-3}.

i) What is the height of the block above mercury level?

ii) Water of density 1000kgm^{-3} is poured into the vessel until it just covers the steel block. What is the height of the water column?

Solution



Volume of steel block $= 12 \times 12 \times 12 = 1728cm^3$

Weight of steel block = $v\rho$g $= (1728 \times 10^{-6}) \times 7800$g

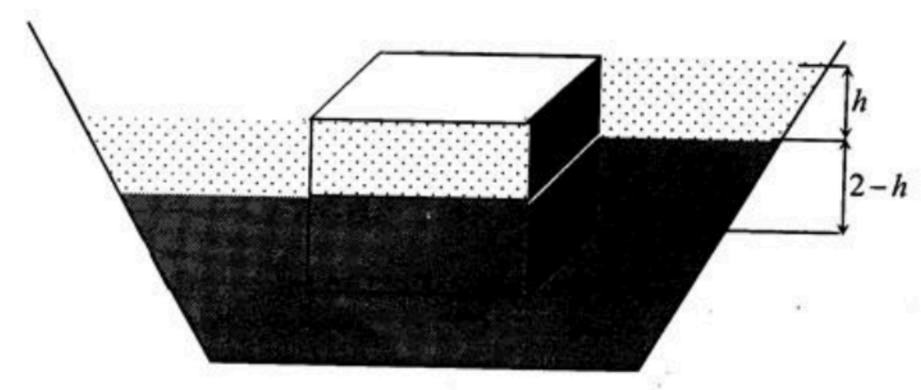
Let h be the height of the block above the mercury surface.

Volume of the mercury displaced $= 12 \times 12 \times (12 - h)$

$\therefore Weight\ of\ mercury\ displaced = 144(12 - h) \times 10^{-6} \times 13600$g

$from\ the\ law\ of\ floatation, weight\ of\ steel\ block = weight\ of\ mercury\ displaced$

$\therefore (1728 \times 10^{-6}) \times 7800g = 144(12 - h) \times 10^{-6} \times 13600g \quad \Rightarrow h = 5.12cm$



Volume of water displaced = $12 \times 12 \times h = 144h \times 10^{-6}\, m^3$

Weight of water displaced = $(144h \times 10^{-6}) \times 1000$g

Volume of mercury displaced = $12 \times 12 \times (12 - h) = 144(12 - h) \times 10^{-6}\, m^3$

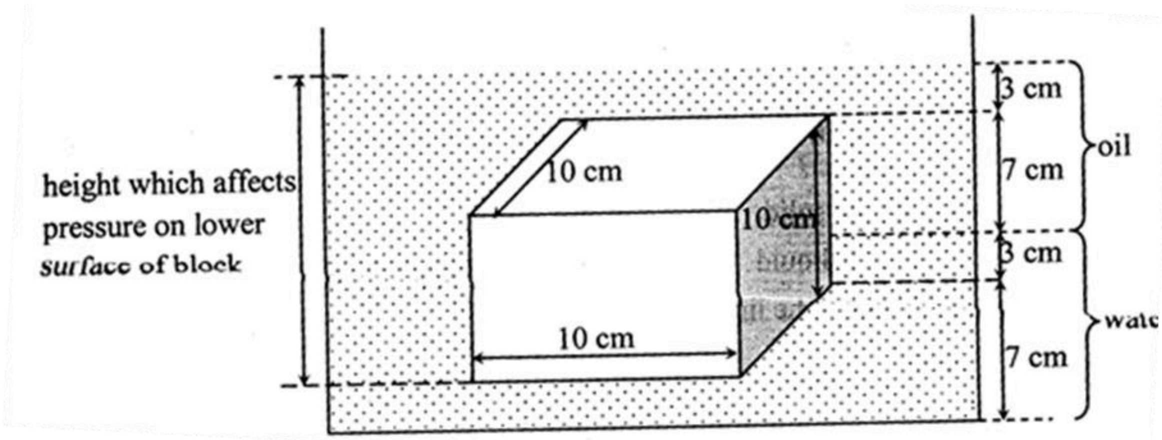
But weight of steel block = weight of water displaced + weight of mercury displaced

$$\therefore (1728 \times 10^{-6}) \times 7800g = (144h \times 10^{-6}) \times 1000g + 144(12-h) \times 10^{-6} \times 13600g$$

$$\therefore 1728 \times 78 = 1440h + 19584(12-h) \Rightarrow h = 0.18cm$$

6. A cubical block of wood 10cm along each side floats at the interface between oil and water with its lower surface cm below the interface. The heights of oil and water columns are 10cm each. The density of oil is 800kgm^{-3} and that of water is 1000kgm^{-3}.
(i) What is the mass and density of the block?
(ii) What is the pressure on the lower surface of the block?

Solution



Note that oil floats on water because water is more dense (has a higher value of density).

Volume of oil displaced = $7 \times 10 \times 10 = 700cm^3$

Mass of oil displaced = $(700 \times 10^{-6}) \times 800 = 5.6 \times 10^{-1}kg$

Volume of water displaced = $3 \times 10 \times 10 = 300cm^3 = 300 \times 10^{-6}m^3$

Total mass of fluid displaced = $5.6 \times 10^{-1} + 3 \times 10^{-1} = 8.6 \times 10^{-1}kg$

But mass of floating body = mass of fluid displaced. $\Rightarrow$ Mass of block = 0.86kg

Density of block = $\frac{mass}{volume} = \frac{0.86}{10\times10\times10\times10^{-6}} = 860kgm^{-3}$

(ii) Pressure due to oil column = hρg $= \frac{10}{100} \times 800 \times 9.81 = 784.8Nm^{-2}$

Pressure due to water column = hρg $= \frac{3}{100} \times 1000 \times 9.81 = 294.3\ Nm^{-2}$

$\therefore$ Total pressure on lower surface = 784.8 + 294.3 = 1079.1 Nm^{-2}

Trial questions

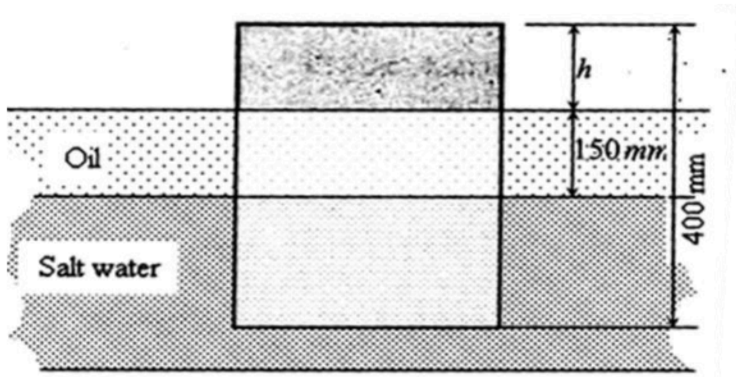
1. A solid weighs 237.5g in air and 212.5g when totally immersed in a liquid of density 0.9g/cc. Calculate the :
(i) density of the solid,

(ii) Density of a liquid in which the solid would float with $\frac{1}{5}$ of its volume exposed above the liquid surface. [Ans. (i) $\rho = 9500kgm^{-3}$ (ii) $\rho = 1190\ kgm^{-3}$]

2. A block of wood of density ρ floats at the interface between immiscible liquids of densities $\rho_1$and ρ_2, such that three quarters of the block's volume is in the liquid of density ρ_1. if the whole block is coverd by the liquids, and given that ρ_2> ρ_1, show that:

$$\frac{\rho_2 - \rho_1}{\rho - \rho_1} = 3$$

3.



In the figure above, a water proof block of wood in form of cube of side 400mm is floating in a tank of salt water with a 150mm layer of oil floating on the water. Given that the density of oil is 900kgm^{-3}, that of salt water 1030kgm^{-3} and that of wood 800kgm^{-3}, calculate the height h of the block above the surface of the oil.

[Ans h = 70.4mm]

4. A block of wood floats in water of density 1000kg m^{-3} with $\frac{2}{3}$ of its volume submerged. In oil it has $\frac{9}{10}$ of its volume submerged. Find the densities of wood and oil.

[Ans: $\rho_{oil} = 740.74$, $\rho_{water} = 666.67kgm^{-3}$]

DENSITY ROD

This is an instrument used to measure the relative density of liquids.

Procedure

Let the cross-sectional area of the density rod be A.

The rod is submerged in a liquid of density ρ_1 and its length, s_1 submerged in the liquid is noted.

The rod is then submerged in water of density ρ_2, and its length, s_2 submerged in the water is noted

When the rod is floating in the liquid: Volume of liquid displaced = As_1

Mass of liquid displaced = A$\rho_1 s_1$

Weight of liquid displaced = A$\rho_1 s_1$g

But from the law of floatation: Weight of liquid displaced is equal to the weight of the floating rod.

∴ Weight of floating rod = A$\rho_1 s_1$g … … … … …. . i)

When the rod is floating in water: volume of liquid displaced = As_2

Mass of liquid displaced $= A\rho_2 s_2$

Weight of liquid displaced $= A\rho_2 s_2 g$

Similarly, weight of floating rod $= A\rho_2 s_2 g$ ……………….. ii)

But equations (i) and (ii) are equal, $\Longrightarrow$ $A\rho_1 s_1 g = A\rho_2 s_2 g$

$$\therefore \frac{\rho_1}{\rho_2} = \frac{s_2}{s_1}$$

But, $\frac{Density\ of\ a\ substance\ (\rho_1)}{Density\ of\ water\ (\rho_2)}$ = Relative density $\Longrightarrow$ Relative density $= \frac{s_2}{s_1}$

The reader should note that this method of determining relative density has an advantage that the relative density can be got even if the densities of the substances are not known.

Example

1. A hydrometer floats in water with of its 72% of its volume submerged. The hydrometer floats in another liquid with 80% of its volume submerged. Find the relative density of the liquid.

Solution

Let the volume of the hydrometer be v, the density of water be ρ_w, and that of the liquid be ρ_l.

While in water: Volume of water displaced $= \frac{72}{100}\ v$

Mass of water displaced $= \frac{72}{100}\rho_w\ v$

Weight of water displaced $= \frac{72}{100}\rho_w\ vg$

From the law of floatation, weight of hydrometer $= \frac{72}{100}\rho_w\ vg$ ……………….. (i)

While in the liquid: Volume of liquid displaced $= \frac{80}{100}\ v$

Mass of liquid displaced $= \frac{80}{100}\rho_1\ v$

Weight of liquid displaced $= \frac{80}{100}\rho_1\ vg$

Similarly, weight of hydrometer $= \frac{80}{100}\rho_1\ vg \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$

Equating equations (i) and (ii) gives: $\frac{72}{100}\rho_w\ vg = \frac{80}{100}\rho_l\ vg$

$$\Longrightarrow \frac{\rho_l}{\rho_w} = \frac{72}{80},\ But\ \frac{\rho_l}{\rho_w} = Relative\ density$$

$$\Longrightarrow\ Relative\ density = 0.9$$

CHAPTER 7: FLUID FLOW

Types of fluid flow

There are two types of fluid flow, turbulent flow and streamline/steady/laminar flow.

Streamline flow

This is a type of fluid flow where the successive particles passing at a certain point have the same velocity. Equidistant layers from the axis have the same velocity, and lines of flow are parallel to the axis of the tube or pipe.

It can be noted that the paths that represent the direction of the velocities of the particles of the fluid are called streamlines or lines of flow.

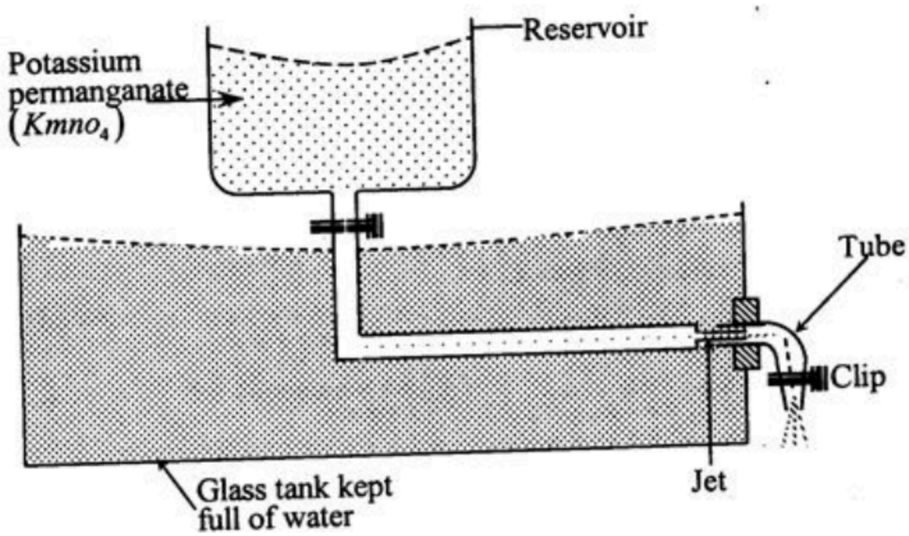
Turbulent flow

This is a type of fluid flow where successive particles passing at a certain point have different velocities, and are disorderly. Equidistant layers from the axis have different velocities, and lines of flow are not parallel to the axis of the tube or pipe.

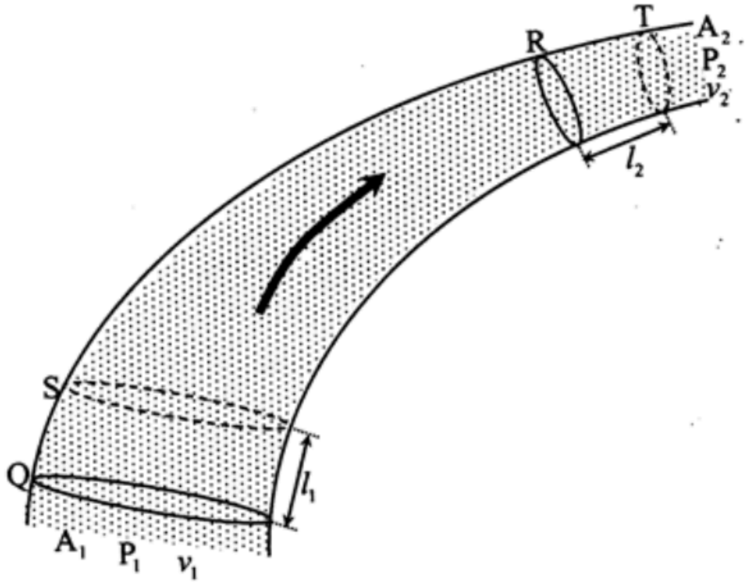
The difference between a **streamline** and a **line of flow** is that a streamline is a curve whose tangent at any point is along the direction of the velocity of the fluid particles at that point. Streamlines never cross. On the other hand, a line of flow is the path followed by a fluid particle. However, in steady flow, the streamlines coincide with the lines of flow.

Experiment to demonstrate streamline flow and turbulent flow

Potassium permanganate solution from a reservoir is fed into the flowing water by a fine jet. The clip is used to control the flow of water along the tube. A fine coloured stream is observed along the centre of the tube for low flow velocities. This type of flow is called stream line. As the rate of flow increases, the coloured stream starts to break up and the colour rapidly spreads throughout the tube. This demonstrates turbulent flow.



Motion of fluid in a tube/pipe of non-uniform cross-sectional area



Consider a fluid such as water of negligible viscosity (Non-viscous) flowing steadily in a tube of non-uniform cross-sectional area in the direction shown.
Also, assume that the velocity at the entrance point, Q is v_1 , the pressure is P_1 , and that the cross sectional area is A_1 , while the velocity at point R is v_2, the pressure is P_2, and that the cross sectional area is A_2 . (Assume that l_1 and l_2 are so small that the parameters A, P and v remain unchanged between the regions QS and RT)
If the fluid moves from QR to ST in a short time Δt, then the volume between Q and S is equal to the volume between R and T.
Volume between Q and S = $A_1 l_1$ and Volume between R and T = $A_2 l_2$

$$\Rightarrow A_1 l_1 = A_2 l_2 \ldots\ldots\ldots\ldots\ldots\ldots i)$$

$$\therefore \frac{A_1}{A_2} = \frac{l_2}{l_1} \text{ and since } A_1 > A_2\text{, then } l_2 > l_1$$

Therefore, since the distances l_1 and l_2 are covered in the same time interval, l_2 must be covered faster than l_1. This implies that implies that $v_2 > v_1$.
Therefore, the fluid flows faster at a narrow part than at a wider part.
It can also be noted that since the volume of fluid entering the tube should be equal to the volume of fluid leaving the tube, then; mass of fluid entering the tube per second is equal to the mass of fluid leaving the tube per second
From Mass= (Volume) ×(density), if the density of the fluid is ρ

Mass entering mass per second $A_1 l_1 \rho \quad \Rightarrow Mass\ per\ second = \frac{A_1 l_1 \rho}{\Delta t}$

Similarly, mass leaving per second = $\frac{A_2 l_2 \rho}{\Delta t}$

$$\therefore \frac{A_1 l_1 \rho}{\Delta t} = \frac{A_1 l_1 \rho}{\Delta t} \Rightarrow A_1 \frac{l_1}{\Delta t} = A_2 \frac{l_2}{\Delta t}$$

But $\frac{distance}{time} = velocity, \Rightarrow \ A_1 v_1 = A_2 v_2 \ldots\ldots\ldots\ldots\ldots (ii)$

It can still be seen from equation ii) that: $\frac{A_1}{A_2} = \frac{V_1}{V_2}, and\ since\ \ A_1 > A_2, then\ v_2 > v_1$

Equations i) and ii) are called equations of continuity.

Therefore, if A is the area and v is the velocity, then Av = constant.

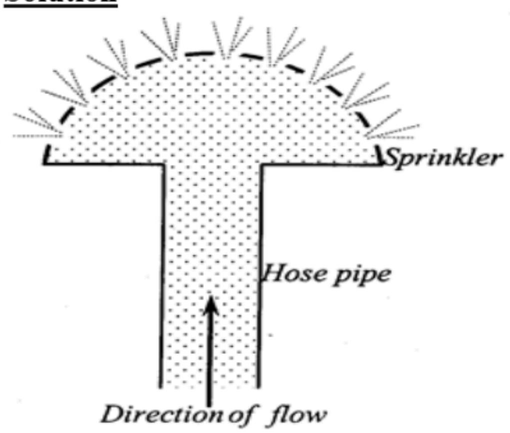
`Av is known as flow rate of volume flux.

It should be noted that the equations are only true for an incompressible fluid, such that its density is constant throughout the tube.
An incompressible fluid is one in which changes in pressure produce no change in the density of the fluid. Liquids and gases are taken to be incompressible.

Example

A lawn sprinkler has 20 holes each of cross-sectional area $2\times 10^{-2}\ cm^2$. the sprinkler is connected to a hose pipe of cross sectional area. If the speed of the water in the hose pipe is $1.5ms^{-1}$, estimate the speed of the water as it emerges from the holes.

Solution



For sprinkler: $A''=2\times 10^{-2}cm^2 = 2\times 10^{-6}\ m^2$

Since it has 20 holes, then; total area $A_s = 20\times 2\times 10^{-6}\ m^2$

$= 40\times 10^{-6}\ m^2$

For hose pipe:

$A_h = 2.4cm^2 = 2.4\times 10^{-4}\ m^2$

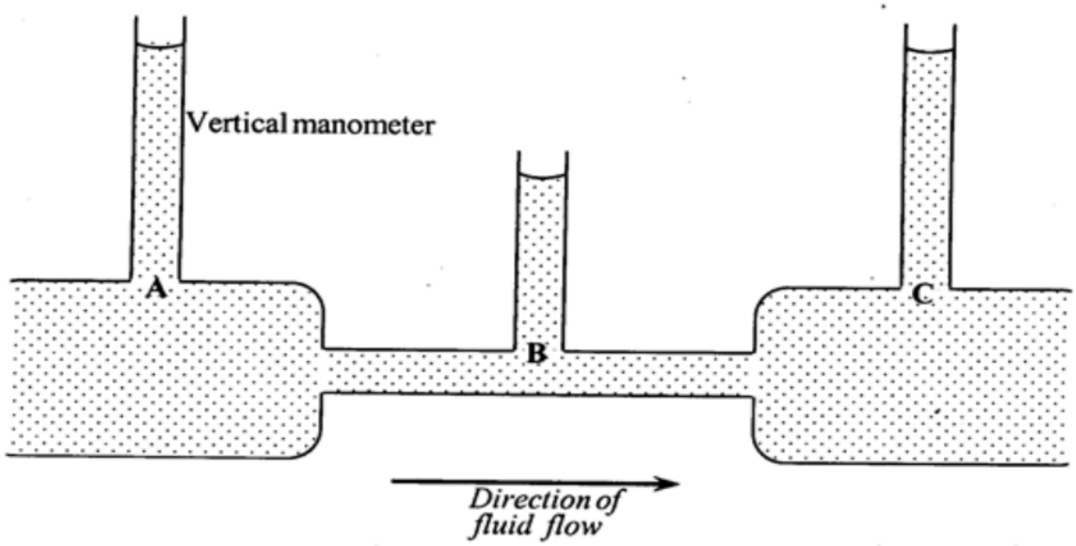
$V_h = 1.5ms^{-1}$

From the equation of continuity, $A_hV_h = A_sV_s$

$\Rightarrow (2.4\times 10^{-4})\times 1.5 = (40\times 10^{-6})\times V_h$

$\therefore V_s = 9ms^{-1}$

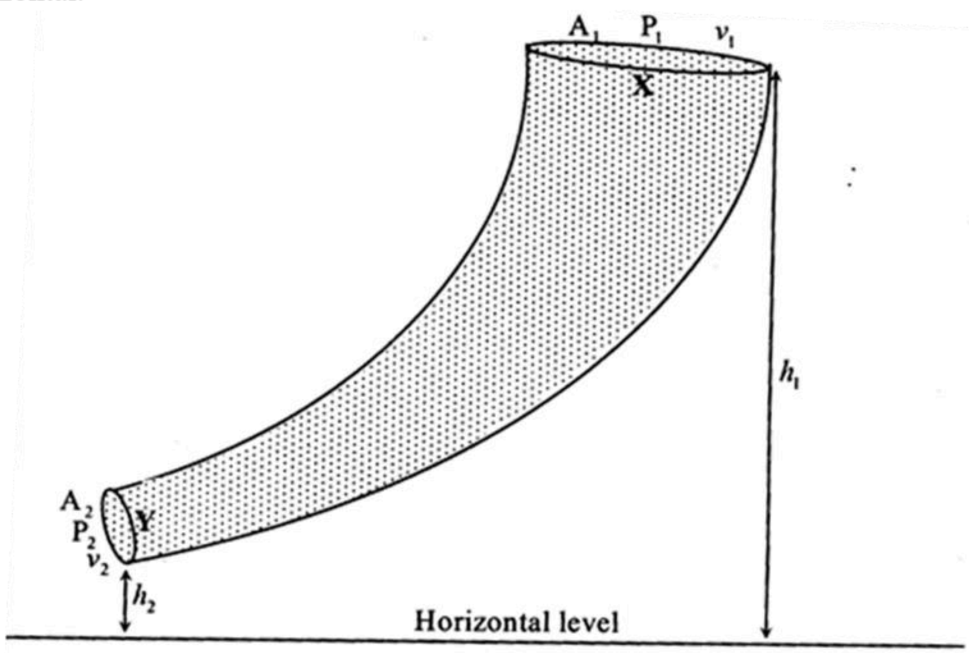
DANIEL BERNOULLI'S PRINCIPLE



When a fluid is at rest, pressure is the same at all points on the same horizontal level. When the fluid is in motion, pressure is not always the same, for example: consider the diagram above; the pressure at the different points is shown by the height of the fluid in the vertical manometers. Pressures are high at parts A and C and falls in part B, but the velocity of the fluid is greatest in the narrow part B, and least in the wider parts A and C. It therefore follows that a decrease in pressure is accompanied by an increase in velocity.

Derivation of Bernoulli's principle

Consider an imaginary tube in a fluid between points X and Y which are at heights h_1 and h_2 from the horizontal.



Let for end X, the pressure be P_1, the cross-sectional area be A_1, and the velocity of fluid be v_1; while at end Y the pressure is P_2, the cross-sectional area is A_2, and the velocity is v_2. Also, consider the cross-sectional area of the tube to be constant at a particular time, for a small time interval Δt.

A fluid in stream line flow has three types of energy

i) Pressure energy,
ii) Potential energy,
iii) Kinetic energy.

Pressure energy is the energy possessed by the fluid by virtue of its pressure at a particular point.

OR

It's the work done by the pressure in moving a fluid through a small displacement.

For end X:

Work done by the pressure P_1 in a short time interval Δt is given by:

Work done = (force)× (distance) but Force = (pressure)× (area)

⟹ Work done = $(P_1A_1)\times(d)$ but distance, d = (velocity) × (time taken)

⟹ Work done = $(P_1A_1)\times(v_1\,\Delta t) = P_1A_1v_1\Delta t$

∴ Pressure energy = $P_1A_1v_1\Delta t$

Kinetic energy $= \frac{1}{2}mv_1^2$

But mass, m = density × volume = $(\rho)\times(A_1l_1)$ $but\ l_1 = v_1\Delta t$

Mass, $m_1 = \rho A_1v_1\Delta t$ ⟹ kinetic energy $= \frac{1}{2}(\rho A_1v_1\Delta t)v_1^2$

Potential energy = mgh

$\therefore\ Potential\ energy = (\rho A_1v_1\Delta t)gh_1$

Therefore total energy at end X is given by:

Pressure energy + Kinetic energy + Potential energy

$$= P_1A_1v_1\Delta t + \frac{1}{2}(\rho A_1v_1\Delta t)v_1^2 + (\rho A_1v_1\Delta t)gh_1$$

Similarly; Total energy at end Y $= P_2A_2v_2\Delta t + \frac{1}{2}(\rho A_2v_2\Delta t)v_2^2 + (\rho A_2v_2\Delta t)gh_2$

If it's assumed that the fluid is:

i) incompressible,
ii) non viscous, and
iii) that the flow is stream lime, then;

From conservation of energy, total energy at **X** should be equal to that at **Y**

$$\therefore P_1A_1v_1\Delta t + \frac{1}{2}(\rho A_1v_1\Delta t)v_1^2 + (\rho A_1v_1\Delta t)gh_1 = P_2A_2v_2\Delta t + \frac{1}{2}(\rho A_2v_2\Delta t)v_2^2 + (\rho A_2v_2\Delta t)gh_2$$

But volume of fluid entering at **X** should be equal to the volume leaving at end **Y**

i.e. $(\rho A_1v_1\Delta t) = (\rho A_2v_2\Delta t)$

$$\therefore P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$$

Since P_1, P_2, v_1, v_2, h_1 and h_2 are randomly chosen, then;

$\Rightarrow P + \frac{1}{2}\rho v^2 + \rho gh$ is a constant

The above equation is called Bernoulli's equation

Therefore, for a pressure P at any part of the tube with a velocity at the same point being v, and the density of the fluid assumed to be constant then Bernoulli's principle can be stated as below,

Bernoulli's principle states that the sum of pressure at any part, the kinetic energy per unit volume and the potential energy per unit volume is always a constant

OR

The total energy of any incompressible and non-viscous fluid in a streamline flow remains constant throughout the flow.

NB. K.E of mass m = $\frac{1}{2}mv^2 \Rightarrow$ K.E per unit mass = $\frac{\frac{1}{2}mv^2}{m} = \frac{1}{2}v^2$

$\therefore$ K.E per unit volume = $\frac{\frac{1}{2}mv^2}{V} = \frac{1}{2}\frac{m}{V}v^2 = \frac{1}{2}\rho v^2$

Also, P.E of mass m = mgh $\therefore$ PE per unit mass = gh $\Rightarrow$ PE per unit volume = ρgh

Example

A fluid of density 1000kgm^{-3} flows in a horizontal tube. If the pressure between the ends of the tube (i.e. at entry and exit) is 10^5 pa and 10^3 pa respectively, and given that the velocity of the fluid at entry is $8ms^{-1}$ calculate the velocity of the liquid at exit.

Solution

From Bernoulli's principle, $P + \frac{1}{2}\rho v^2 + \rho gh =$ constant

$$\Rightarrow P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$$

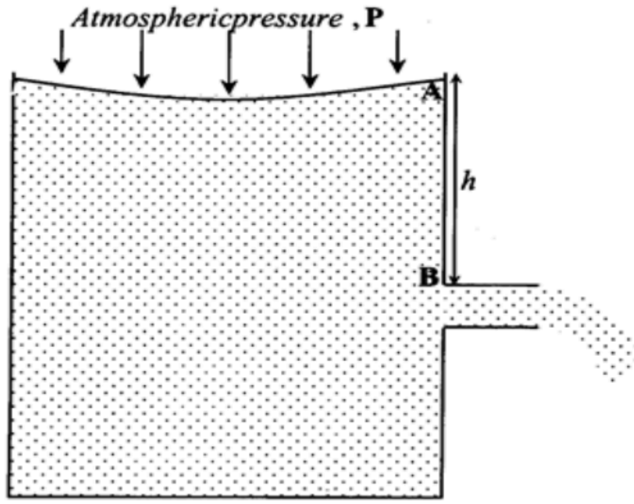
But since the tube is horizontal is horizontal, $h_1 = h_2$

$$\therefore P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow 1000000 + \frac{1}{2} \times 1000 \times 8^2 = 10000 + \frac{1}{2} \times 1000 \times v_2^2$$

$$\therefore v_2 = 45.2\ ms^{-1}$$

Flow of a fluid from a wide tank



From Bernoulli's equation, $P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$

$$P + \frac{1}{2} \times \rho \times 0^2 + \rho g h_1 = P + \frac{1}{2} \times \rho \times v_2^2 + \rho g \times 0$$

$$\Rightarrow \frac{1}{2}v^2 = gh \quad \therefore \ v = \sqrt{2gh}$$

Example

An open tank holds water 1.25m deep. As small hole of cross sectional area $3cm^2$ is made at the bottom of the tank. Assuming that the density of water is $1000kgm^{-3}$, calculate the mass of water per second initially flowing out of the hole.

Solution

Mass = (volume) × (density) ⟹ mas per second = (volume per second) × (density)
But volume = (area) × (distance) ⟹ volume per second = (area) × (distance per second)
But distance per second, $= \frac{distance}{time}$ = velocity
∴ Volume per second = (area) ×(velocity)
⟹ Mass per second (area) × (velocity) × (density)
Velocity = $\sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.25} = 4.95ms^{-1}$
∴ Mass per second = $(3 \times 10^{-4}) \times 4.95 \times 1000 = 1.49kgs^{-1}$

Trial questions

1. Water stands at a depth at depth H in a vertical tank. A hole is made in one of the walls at a depth h below the water surface. The emerging stream of water strikes the floor at a distance from the tank. If the stream of water takes a time t to strike the floor, show that:

   i) $t = \sqrt{\frac{2(H-h)}{g}}$ ii) $R = 2\sqrt{h(H-h)}$

2. A tank empties through a length of horizontal capillary tubing inserted near its base. After being filled with water, the tank empties in 100 seconds. When the water is replaced with

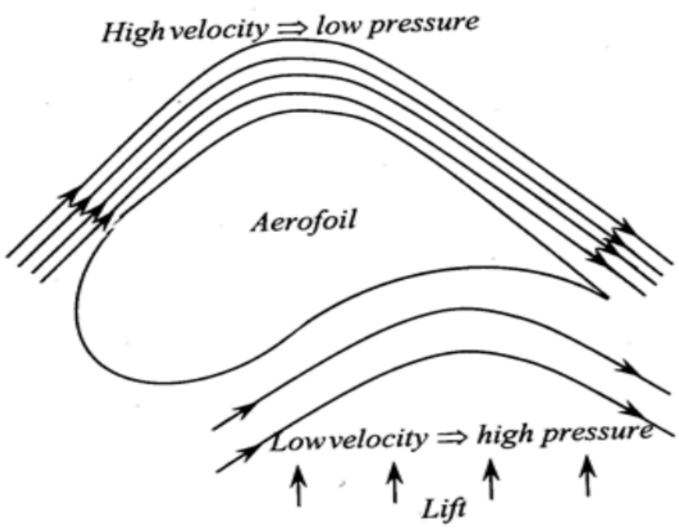
another liquid Y, it takes 165 seconds for the tank to empty. If the relative density of liquid Y is 0.87, and the coefficient of viscosity of water is $1.2\times 10^{-3}Nm^{-2}$, find the coefficient of viscosity of liquid Y.(Read the section about viscosity first before attempting this question)

APPLICATION OF BERNOULLI'S PRINCIPLE

1. Consider a thin sheet of paper held at one end, such that it's horizontally below the lips, with the other end sagging under its own weight.
On blowing steadily over the top of the paper, the sagging end of the paper rises. This is due to the fact that the air above the sheet of paper is higher than that under it. From Bernoulli's principle, this implies that the pressure of air below is greater than that above, and so there is a resultant upward force on the paper which provides the lift.

2. Consider a person standing close to a rail way line.
The air between the person and a moving train has a higher velocity than that behind him. It therefore follows that the pressure in front of the man is less than that behind him. This causes a 'pull' on the man towards the train

3. Aero foil lift



The curved shape of the aero foil creates a faster flow of air over its top surface as compared to the lower one. From Bernoulli's principle, the pressure of air below is greater than that above, and so there is a resultant upward force on the aero foil. This produces a lift on the aero foil. The aero foil (and therefore the lift produced) is applied on the wings of an aero plane at takeoff.

Example

A particular air craft design calls for a dynamic lift of 2.4×10^{4} N on each square metere of the wing when the speed of the air craft through the air is $80ms^{-1}$. Assuming that the air flows past the wing with streamline line flow and that the flow past the lower surface is equal to the speed of the air craft, what is the required speed of the air over the upper surface of the wing? (Assume that density of air is 12.9kgm^{-3})

Solution

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Where, P_1 is the pressure on upper part, P_2 pressure on lower part, v_1 the velocity of air on upper part, v_2 velocity of air on lower surface and A = 1m^2 the area of the wing

$$h_1 = h_2 = h \Rightarrow P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore P_2 - P_1 = \frac{1}{2}\rho(v_1^2 - v_2^2) = \frac{1}{2} \times 1.29 \times (v_1^2 - 80^2)$$

But; dynamic pressure (force of dynamic lift) = (pressure difference) × (surface are of wing)

$$\therefore 24000 = [\frac{1}{2} \times 1.29 \times (v_1^2 - 6400)] \times 1 \Rightarrow v_1 = 208.8ms^{-1}$$

Trial questions

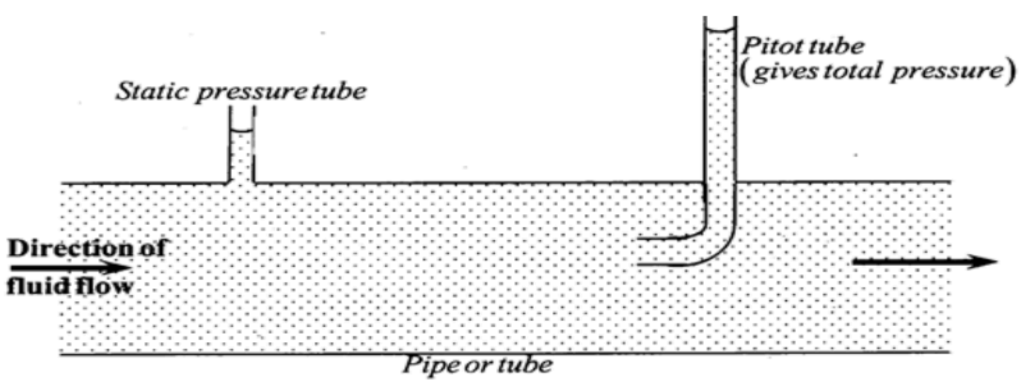
In his test experiment on a model aero plane in a wind tunnel, Rashid found that the flow speeds on the upper and lower surfaces of the wing are 81ms^{-1} and 57ms^{-1} respectively. If wing surface area is 3.2m^2 and density of air is 1.3kgm^{-3} , find the dynamic force on the wing.

Ans. 6.9kN (2 sf)

4. FLOW METERS

These are devices used to measure the rate of flow a fluid i.e. fluid velocity, or volume per second of a fluid through a pipe.

- Pitot-static tube



The total pressure exerted by a flowing liquid has two components; static pressure, which is the pressure it would have if it were at rest, and dynamic pressure, which is the pressure due to its velocity. The total pressure is measured by the Pitot tube, while the manometer connected at right angles to the pipe (static tube) measures static pressure.

From Bernoulli's principle, $P + \frac{1}{2}\rho v^2 + \rho g h =$ constant

Static pressure and dynamic pressure = $P + \rho gh, and\ dynamic\ pressure\ \frac{1}{2}\rho v^2$

Static pressure + Dynamic pressure = Total pressure

$\Longrightarrow$ Dynamic pressure = Total pressure - Static pressure

$\therefore\ \frac{1}{2}\rho v^2$ = Total pressure - Static pressure

$$\Longrightarrow\ v = \sqrt{\frac{2}{\rho} \times (\,Total\ pressure - Static\ pressure)}$$

The reader should at this point therefore be able to describe an experiment to determine the velocity of water in a uniform pip**e**

Example

Water flows steadily along a uniform flow tube of cross sectional area 30cm^2 . The static pressure is 1.2$\times 10^5 pa$ and the total pressure is 1.28$\times$ 10^5 pa. Assuming that the density of water is 1000kgm^{-3}, calculate the:

i) flow velocity,
ii) volume flux,
iii) mass of water passing through a section of the tube per second.

Solution

i) $v = \sqrt{\frac{2}{\rho} \times (\,Total\ pressure - Static\ pressure)}$

$v = \sqrt{\frac{2}{1000} \times (1.28 - 1.20) \times 10^5}\ \ \Longrightarrow\ \ v = 4ms^{-1}$

ii) Volume flux = volume per second = (area) $\times$ (velocity)

$= (\,30 \times 10^{-4}) \times 4$

$= 0.012$

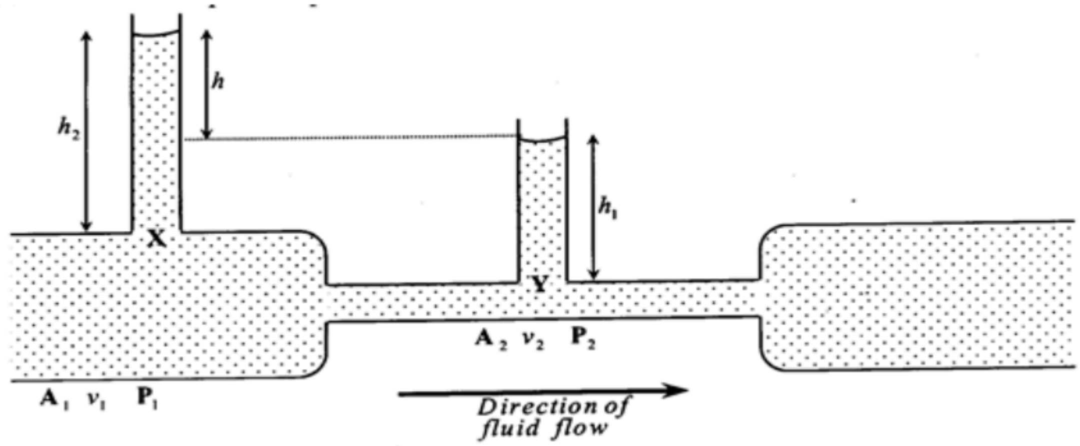
iii) Mass per second = (volume per second) $\times$ (density)

$= 0.012 \times 1000$

= 12kgs^{-1}

- **Venturimeter**

This is a horizontal tube with a constriction at one part. The tubes X and Y measure the pressures at the respective parts of the tube.



From Bernoulli's principle, $P + \frac{1}{2}\rho v^2 + \rho g h =$ constant

$\Rightarrow P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$

But since the tube is horizontal $h_1 = h_2$

$\therefore P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

$$\Rightarrow P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = \frac{1}{2}\rho(v_2^2 - v_1^2) \ldots\ldots..i)$$

For incompressible fluid, $A_1V_1 = A_2V_2 \Rightarrow v_1 = \frac{A_2}{A_1}v_2$

Substituting for $v_1$in equation i) gives: $P_1 - P_2 = \frac{1}{2}\rho(\, v_2^2 - (\frac{A_2}{A_1}v_2)^2)$

$$\therefore P_1 - P_2 = \frac{1}{2}\rho v_2^2\left(1 - \frac{A_2^2}{A_1^2}\right)$$

But $P_1 - P_2 = h\rho g$, $\therefore h\rho g = \frac{1}{2}\rho v_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] \Rightarrow v_2 = \sqrt{\frac{2gh}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]}}$

Once, v_2 is determined, v_1 can be determined by substitution, and hence volume per second $A_1 v_1$ or A_2v_2or can be found. Getting an equivalent v_1 *is* left as an exercise to the student.

Example

Water flows along a horizontal pipe of cross-sectional area cm which has a constriction of cross - sectional area 12cm^2 at one part. If the speed of the water at the constriction is 4ms^{-1},

i) Calculate the speed of water at the wider section.

ii) Given that the pressure at the wider section is 1.0×10^5 pa, and that the density of water is 1000kgm^{-3}, calculate the pressure at the constriction.

Solution

i) $A_1\, v_1 = A_2 v_2 \Rightarrow v_1 = \frac{A_2}{A_1} v_2 = \frac{12}{48} \times 4 = 1ms^{-1}$

The reader should note that there is no need to first convert the cross-sectional areas to SI units since we are taking their ratios, otherwise they would have been converted first.

i) From Bernoulli's principle, $P + \frac{1}{2}\rho v^2 + \rho g h =$ constant

$$\Rightarrow P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

But since the tube is horizontal is horizontal, $h_1 = h_2$

$\therefore P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

$\Rightarrow P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$

$\therefore 100000 - P_2 = \frac{1}{2} \times 1000 \times (4^2 - 1^2) \Rightarrow P_2 = 9.25 \times 10^4 pa$

Trial question

Oil of density 800kgm^{-3} flows steadily through a horizontal pipe of non-uniform cross section. If the pressure of oil is 7cm of mercury column at a point where the velocity of flow is 60cms^{-1}, what is the pressure at another point were the velocity of flow is 85cms^{-1}. (Given: density of mercury is 13600kgm^{-3}) [Ans: 6.891cm of mercury column]

VISCOSITY

This is the resistance within the fluids. Liquids that pour easily are more viscous than those that pour very fast. For example engine oil, syrup, glue, etc. are more viscous as compared to water, milk paraffin etc.
The viscosity of a liquid also affects the motion of a solid through it, for example if different liquids are put in different measuring cylinders and a small bearing made to fall through each liquid, the ball falls faster through a liquid of lower viscosity.

Explanation of viscosity

For a flowing fluid, molecular layers in contact with the sides of the tube are practically stationary because of the attraction between the molecules of the tube and those of the fluid. (Adhesive forces). The successive layers towards the center must therefore slide over one another against the attraction between the molecules of the individual layers (cohesive forces). This effect results into layers towards the center of the tube moving faster than those towards the sides of the tube.
Since the velocities of the neighboring layers are different, a frictional force occurs between the various layers of the fluid.
The frictional force is directly proportional to the area of the molecular layers and to the velocity gradient. This is **Newton's law of viscosity**.
If v_1 and v_2 are velocities of neighbouring molecular layers a distance h apart, their velocity gradient is given by; $\frac{Change\ in\ Velocity}{Distance\ of\ seperation} = \frac{v_2 - v_1}{h}$

Therefore, velocity gradient is the change in velocity between molecular layers of a fluid separated by a distance of one metre.
The SI unit of velocity gradient is $\frac{ms^{-1}}{m} = s^{-1}$
From Newton's law of viscosity, $F \propto A,\ \ and\ \ F \propto \frac{v_2 - v_1}{h}$
It can be noted that liquids that obey the two conditions above are called Newtonian liquids
Combining the two conditions gives; $F \propto A\frac{v_2 - v_1}{h}$

$$\Rightarrow F = \eta\left(A\frac{v_2 - v_1}{h}\right) \text{.................(i)}$$

Where η is a constant of proportionality known as the coefficient of viscosity. The value of this constant depends on the nature of the fluid

Definition

From equation (i) $\eta = \frac{F}{A\left(\frac{v_2 - v_1}{h}\right)}$
Therefore, coefficient of viscosity is the frictional force acting on an area of $1m^2$ of a fluid when it's in a region of velocity gradient $1s^{-1}$.

Alternatively;

Coefficient of viscosity can be defined as the tangential stress which layer of a fluid exerts on another layer in contact with it when the velocity gradient between the layers is $1s^{-1}$
The SI unit of $\eta = \frac{N}{m^2 s^{-1}} = Nsm^{-2}$

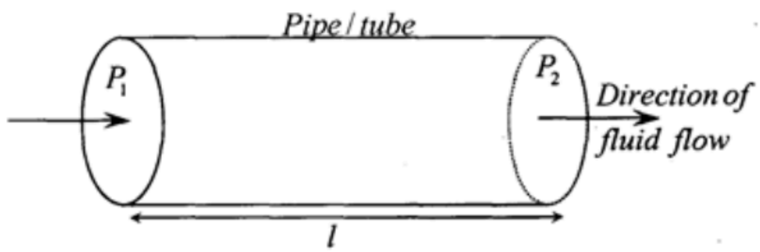
Dimensions of η:

$$[\eta] = \frac{[Force]}{[Area]\times\left[\frac{v_2-v_1}{h}\right]} = \frac{MLT^{-2}}{L^2\times\frac{LT^{-1}}{L}} = ML^{-1}T^{-1}$$

Note: (i) Viscosity in liquids is due to molecular attraction between molecules of neighbouring layers and therefore energy is required to drag one layer over the other.

(ii) Viscosity in liquids decreases with increase in temperature. This is due to the fact that the viscosity in liquids is due to intermolecular forces of attraction such that as temperature increases, the intermolecular forces are broken down, and so the molecules travel faster and further apart.

POISSEULLES FORMULAR FOR FLUID FLOW



Consider a streamlined flow of a liquid of coefficient of viscosity η in a horizontal tube, of radius, *a,* length *l* and cross sectional area A, with the ends of the tube maintained at pressures P_1 and P_2.

If P is the pressure difference then; $P = P_1 - P_2$

Pressure gradient $= \frac{P_1-P_2}{l} = \frac{P}{l}$

Therefore, pressure gradient is the change in pressure of a fluid in a tube of length one metre.

The volume of liquid flowing out of the tube per second depends on

- The coefficient of viscosity, η
- Radius of the tube , a
- Pressure gradient, $\frac{P}{l}$

Volume per second $\propto \eta$, volume per second $\propto a$a volume per second $\propto \frac{P}{l}$

Combining the expressions gives: volume per second $\propto \eta a \frac{P}{l}$

$\Rightarrow$Volume per second $= k\left(\eta a \frac{P}{l}\right)$

Where k is a constant of proportionality

Using dimensions

$$[Volume\ per\ second] = [\eta]\times[a]\times\left[\frac{P}{l}\right]$$

$$\Rightarrow L^3T^{-1} = [ML^{-1}T^{-1}]^x \times [L]^y \times [ML^{-2}T^{-2}]^z$$

Equating corresponding indices gives:

$3 = -x + y - 2z$, $-1 = -x - 2z$, $0 = x + z$

Solving the three equations simultaneously gives:

$z = 1$, $x = -1$, and $y = 4$

Therefore, Volume per second = $k\eta^{-1}a^4\left(\frac{P}{l}\right)^1 = \frac{ka^4P}{\eta l}$

Other experiments show that $k = \frac{\pi}{8}$

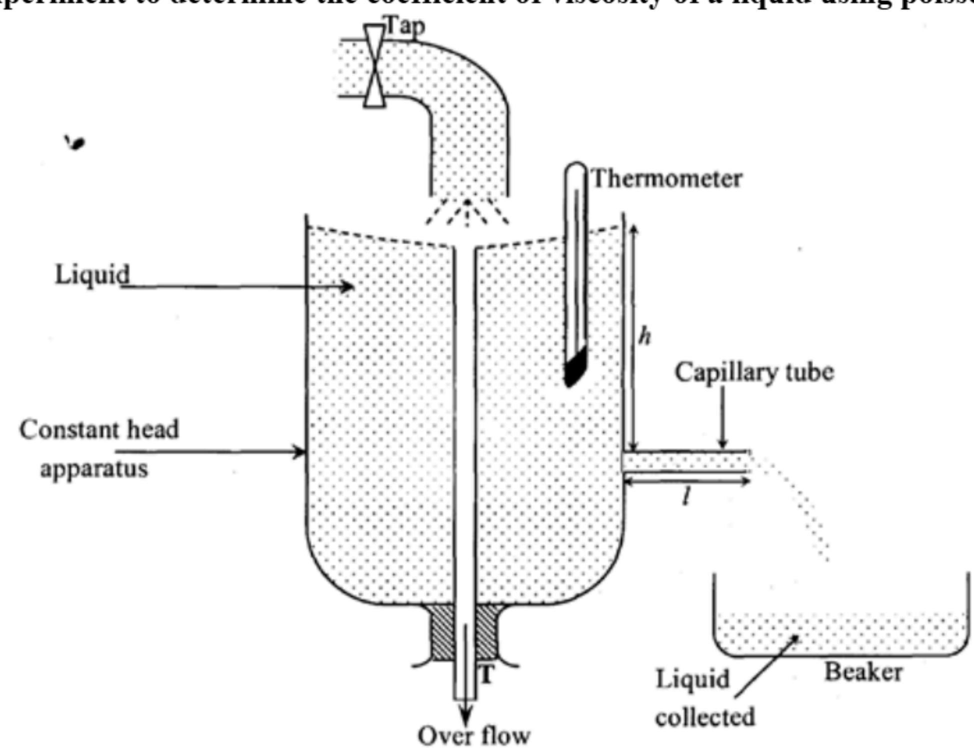
$\Rightarrow volume\ per\ second = \frac{\pi a^4 P}{8\eta l}$

This is known as poisseulle's formula for fluid flow

Note: (i) The formula only holds for laminar/steady flow and hence not very fast flow

(ii) The radius can be *r* instead of *a* in which case the equation changes correspondingly

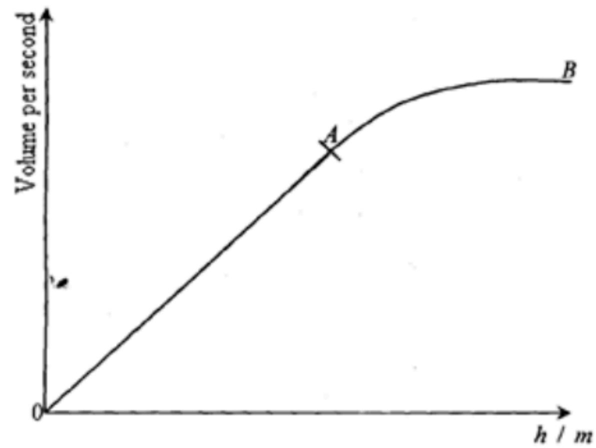
Experiment to determine the coefficient of viscosity of a liquid using poisseulle's formula



The liquid passes from a constant head apparatus through a capillary tube and the volume of liquid collected in a beaker in a certain time (measured using a stop clock) is measured. The height *h* can be changed by altering the position of the tube T. for different heights, h of the capillary tube below the top level of the liquid in the constant head apparatus, the volume of the liquid flowing into the beaker in a given time is determined.

The results are tabulated including values of volume per second

A graph of volume per second against the height h is plotted, and its looks like the one shown.



From Poisseulle's formular,

Volume per second $= \frac{\pi a^4 P}{8\eta l}$

But pressure, $P = h\rho g$

$\Rightarrow$ Volume per second $= \frac{\pi a^4 (h\rho g)}{8\eta l}$

$= \frac{\pi a^4 \rho g}{8\eta l} h$

The equation is in the form $y = mx$; where the slope m of the linear part is given by; $\frac{\pi a^4 P}{8\eta l}$

$$\Rightarrow \eta = \frac{\pi a^4 P}{8ml}$$

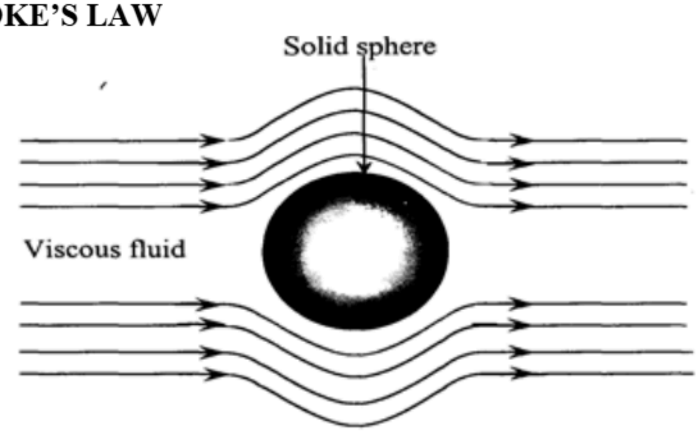
a, the radius of the capillary tube is measured using a micrometer screw gauge, the height h and length l can also be measured, and if the density of the liquid is known, the coefficient of viscosity can be calculated.

Note: (i) The temperature from the thermometer should be constant throughout the experiment

(ii) Liquids which are obtainable in large quantities, and which can flow easily such as water are suitable for this method,

(iii) The constant head apparatus is to ensure that the rate of liquid flowing through the capillary tube is steady/uniform

STOKE'S LAW



Consider a solid sphere moving through a viscous fluid as shown above. As it moves, there is a retarding force, F called viscous drag/viscous force, which opposes its motion in the fluid

Viscous drag is the force that opposes the relative motion of a body in a fluid.

The viscous drag depends on:

- Coefficient of viscosity, η
- Size of body $(radius, r)$

- Velocity of the body, v

  $\therefore F \propto \eta^x v^y r^z \Rightarrow F = k\eta^x v^y r^z$

  Where k is a constant of proportionality

Using dimensions; $[F] = [\eta]^x \times [v]^y \times [r]^z$

$\Rightarrow MLT^{-2} = (ML^{-1}T^{-1})^x \times (LT^{-1})^y \times (L)^z$

Equating corresponding components gives;

$x = 1,\ 1 = -x + y + z\ , \qquad 2 = y + z$

Solving the three equations simultaneously gives;

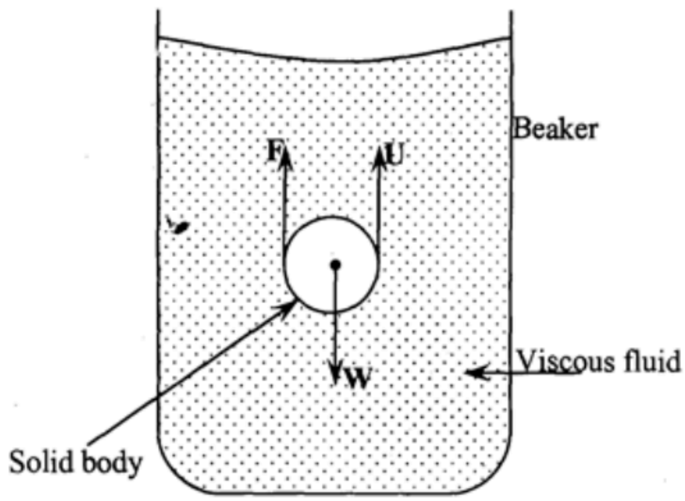
$x = 1,\ \ y = 1\ \ ahd\ z = 1$

Other experiments show that k = 6π

$\therefore F = 6\pi\eta vr$ This is called Stoke's law

Vertical motion of a body in a viscous fluid

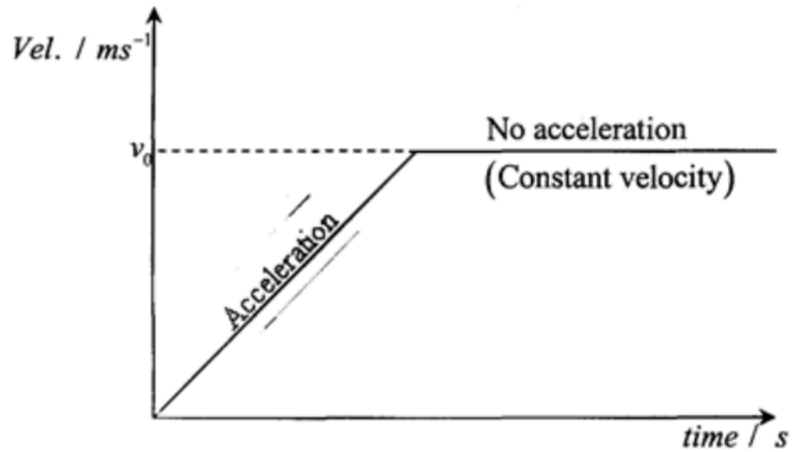
Consider a sphere falling vertically in a viscous fluid in a beaker as shown. At any time, three forces act on it; its weight acting downwards up thrust due to the weight of liquid displaced acting upwards and viscous drag acting upwards.



As the body accelerates downwards its velocity increases and from F = $6\pi\eta vr$, the viscous drag also increases. However, since the body is already completely immersed in the liquid, up thrust remains constant (since no more fluid is being displaced)

A point is reached when W = F + U. This implies that the net force acting on the body is zero. The body therefore continues to move down the fluid with a constant velocity called terminal velocity. Therefore, **terminal velocity** is the maximum velocity attained by a body when falling through a viscous fluid.

Velocity time graph for a body falling through a viscous fluid



At terminal velocity, W = F + U ………..(i)
If density of body = ρ , density of fluid = σ, terminal velocity = v_0, coefficient of viscosity for the fluid = η and the radius of the body is r,

Then, $F = 6\pi\eta\, v_0 r$ (viscous force at terminal velocity)

$W = mg = \mathrm{v}\rho g$, but $v = \frac{4}{3}\pi r^3$, $\Rightarrow W = \frac{4}{3}\pi r^3 \rho g$

Up thrust, U = weight of fluid displaced = $\frac{4}{3}\pi r^3 \sigma g$

Substituting for W, F, and U in equation (i) gives;

$$\frac{4}{3}\pi r^3 \rho g = 6\pi\eta\, v_0 r + \frac{4}{3}\pi r^3 \sigma g$$

$$\Rightarrow \frac{4}{3}\pi r^3 (\rho - \sigma) g = 6\pi\eta\, v_0 r$$

$$\frac{4}{3} r^2 (\rho - \sigma) g = 6\eta\, v_0$$

$$\therefore \eta = \frac{2gr^2}{9v_0}(\rho - \sigma) \quad or\ v_0 = \frac{2gr^2}{9\eta}(\rho - \sigma)$$

Example

27 spherical rain drops of the same mass and radius are falling down with a terminal velocity of $15cms^{-1}$. If they coalesce to form a big drop, what will be its terminal velocity? (Neglect the buoyancy due to air).

Solution

Let the radius of the big drop be R, while that of each of the 27 small droplets be r

Volume of each droplet = $\frac{4}{3}\pi r^3$

Volume of 27 droplets = $27 \times \left(\frac{4}{3}\pi r^3\right)$

Volume of big drop = $\frac{4}{3}\pi R^3$

From conservation of volume (i.e since volume remains unchanged),

$$\frac{4}{3}\pi R^3 = 27 \times \left(\frac{4}{3}\pi r^3\right)$$

$$\therefore R^3 = 27r^3 \Rightarrow R = 3r$$

But terminal velocity = $\frac{2gr^2}{9\eta}(\rho - \sigma)$ but since buoyancy due to air is negligible, $\sigma = 0$

$\Rightarrow$Terminal velocity = $\frac{2gr^2\rho}{9\eta}$

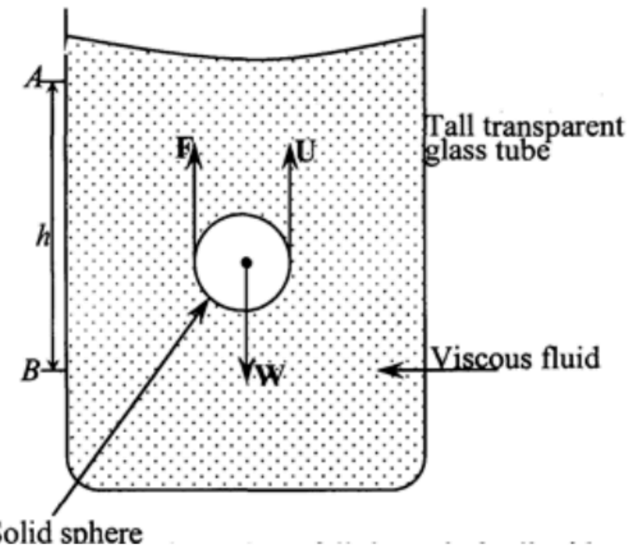
$\therefore$ For small droplets, $v_1 = \frac{2gr^2\rho}{9\eta} = 15$cms^{-1} and for big droplets, $v_2 = \frac{2gR^2\rho}{9\eta}$

Thus $\frac{v_2}{v_1} = \frac{R^2}{r^2} = \frac{(3r)^2}{r^2} = 9 \Rightarrow v_2 = 9v_1$

But $v_1 = 15 \Rightarrow v_2 = 15 \times 9 = 135$cms^{-1} = 1.35ms^{-1}

Experiment to determine the coefficient of viscosity using Stoke's law

A fluid/liquid of known density,σ is put in a tall transparent glass tube. Two reference marks A and B are put on the tube and the distance, h between them is measured



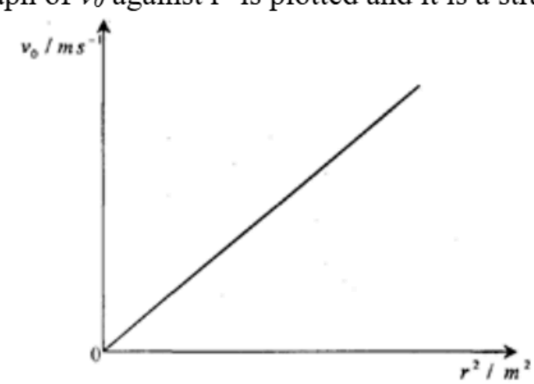
A solid sphere of known density,ρ is made to fall through the liquid and the time, t taken while it falls between the marks A and B is measured. Terminal velocity, $v_0 = \frac{h}{t}$

The radius r of the sphere is measured using a micrometer screw gauge.

The coefficient of viscosity η can be calculated from: $\eta = \frac{2gr^2}{9v_0}(\rho - \sigma)$

However for practical purposes, the experiment is repeated with different solid spheres of different radii, and the corresponding terminal velocities, v_o determined. The results are tabulated including values of r^2.

A graph of v_0 against r^2 is plotted and it is a straight line through the origin as shown



From $v_0 = \frac{2g(\rho - \sigma)}{9\eta} r^2$, the equation is in the form $y = mx$, where the slope, $m = \frac{2g(\rho-\sigma)}{9\eta}$.

$\therefore \eta = \frac{2g(\rho-\sigma)}{9m}$

Alternatively, since $r = \frac{d}{2}$, $then\ v_0 = \frac{2g(\rho-\sigma)}{9\eta}\left(\frac{d}{2}\right)^2 \Rightarrow v_0 = \frac{g(\rho-\sigma)}{18\eta}d^2$
Where d is the diameter of the solid sphere
Therefore, a graph of v_0 against d^2 can the plotted, and the corresponding slope, m determined

Trial question
The table below gives the times of fall of steel spheres of different diameter falling through a distance of 50cm in a viscous liquid of density 1260 kgm^{-3}

Diameter/ mm	2.0	2.2	2.4	2.6
Time/s	8.36	6.89	5.80	4.93

If the density of steel is 7800kgm^{-3}, plot a suitable graph and from it determine the coefficient of viscosity of the liquid.
(Hint: start by calculating the terminal velocities, v_0 for each steel sphere, and then plot a graph of v_0 against d^2) [Ans: $\eta = 0.246Nsm^{-2}$]

Examples
1. A spherical rain drop of radius 2.0×10^{-4}m falls vertically in air at 20^0C. If the densities of air and water are1.2kgm^{-3} and 1000kgm^{-3} respectively, and that the coefficient of viscosity of air at 20^0C is 1.8×10^{-5} pa s, calculate the terminal velocity of the drop.

Solution

$$v_0 = \frac{2g(\rho-\sigma)}{9\eta}r^2 \text{ but } r = 2.0 \times 10^{-4}\text{m}, \rho = 1000,\ \sigma = 1.2, \eta = 1.8 \times 10^{-5}$$

$$\Rightarrow v_0 = \frac{2\times(2\times10^{-4})^2\times9.81}{9\times(1.8\times10^{-5})} \times (1000 - 1.2) = \frac{0.000784}{0.000162} = 4.84\text{ms}^{-1}$$

2. A metal sphere of radius 3.0×10^{-3}m and mass 4.0×10^{-4}kg falls under gravity, centrally down a wide tube filled with a liquid at room temperature (25^0C). The density of the liquid is 800kgm^{-3}. The sphere attains a terminal velocity of 45cms^{-1}. The tube is emptied and filled with another liquid of density 1000kgm^{-3} at the same temperature. When the metal sphere falls centrally down the tube, it's found to attain a terminal velocity of 20cms^{-1}. Determine the ratio of the coefficient of viscosity of the second liquid to that of the first liquid at room temperature.

Solution

For the metal sphere, r = 3.0×10^{-3}m and m = 4.0×10^{-4}kg

$$\text{Volume} = \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times(3.0\times10^{-3})^3 = 1.13\times10^{-7}m^3$$

But density, $\rho = \frac{Mass}{Volume} = \frac{4\times10^{-4}}{1.13\times10^{-7}} = 3540kgm^{-3}$

$$\eta = \frac{2gr^2}{9v_0}(\rho - \sigma)$$

Consider the first liquid: $v_0' = 45 \times 10^{-2}ms^{-1}$, $\sigma' = 800$, $\eta' = ?$

$$\eta' = \frac{2gr^2}{9v_0'}(\rho - \sigma') \ldots\ldots\ldots\ldots\ldots (i)$$

Consider the second liquid: $v_0'' = 20 \times 10^{-2}ms^{-1}$, $\sigma'' = 10000$, $\eta'' = ?$

$$\eta'' = \frac{2gr^2}{9v_0''}(\rho - \sigma'') \ldots\ldots\ldots\ldots\ldots (ii)$$

Dividing the two equations gives;

$$\frac{\eta''}{\eta'} = \frac{\frac{2gr^2}{9v_0''}(\rho-\sigma'')}{\frac{2gr^2}{9v_0'}(\rho-\sigma')} = \frac{v_0'(\rho-\sigma')}{v_0''(\rho-\sigma'')}$$

$$= \frac{(45\times10^{-2})\times(3540-1900)}{(20\times10^{-2})\times(3540-800)} = \frac{1143}{548}$$

$$\therefore \frac{\eta''}{\eta'} = 2.09$$

3. A steel sphere of diameter 3×10^{-3}m falls through a cylinder containing a liquid *x*. When the sphere has attained a terminal velocity, it takes 10.8 seconds to travel between two fixed marks on the cylinder. When the experiment is repeated using another steel sphere of diameter 5×10^{-3}, with the cylinder containing liquid y, the time of fall between the fixed points is 4.8 seconds. If the density of liquid *x* is 1.26×10^3kgm^{-3}, that of liquid y is 0.92×10^3kgm^{-3} and that of steel is 7.8×10^3kgm^{-3}, determine the ratio of the coefficients of viscosity of liquid *x* to that of liquid y, if the temperature remains constant throughout.

Solution

Let the distance between the two reference, marks on the cylinder be h

For liquid *x*, terminal velocity $v_x = \frac{h}{10.8}$ and radius $r_x = \frac{3\times10^{-3}}{2} = 0.0015$m

For liquid y terminal velocity $v_y = \frac{h}{4.8}$ $, and\ radius\ r_y = \frac{5\times10^{-3}}{2} = 0.0025m$

$$\text{From } \eta = \frac{2gr^2}{9v_0}(\rho-\sigma)$$

For liquid x :

$$\eta_x = \frac{2\times9.81\times(0.0015)^2}{9\times\left(\frac{h}{10.8}\right)}(7.8\times10^3 - 1.26\times10^3)$$

$$\Rightarrow \quad \eta_x = \frac{0.34645}{h} \quad \ldots\ldots\ldots\ldots(i)$$

For liquid y:

$$\eta_y = \frac{2\times9.81\times(0.0025)^2}{9\times\left(\frac{h}{4.8}\right)}(7.8\times10^3 - 0.92\times10^3)$$

$$\Rightarrow \quad \eta_y = \frac{0.44995}{h} \quad \ldots\ldots\ldots\ldots(ii)$$

Dividing the two equations gives;

$$\frac{\eta_x}{\eta_y} = \frac{0.34645}{h} \times \frac{h}{0.44995}$$

$$\therefore \frac{\eta_x}{\eta_y} = 0.77$$

Viscosity in gases

Viscosity in gases is due to momentum transfer between the neighbouring layers of gases. The viscosity in gases is directly proportional to the average speed of the gas molecules, and since the average speed of the gas molecules increases with increasing temperature viscosity in gases increases with increasing temperature.

CHAPTER 8: MECHANICAL PROPERTIES OF MATTER

Mechanical properties of materials are concerned with the behavior of such material under the action of forces. Such properties include strength, brittleness, toughness, ductility, stiffness, elasticity etc.

Strength is the ability of a material to withstand or to resist an applied force before such a material breaks. Examples of materials that are strong are metals.

Stiffness is the ability of a material to resist a change in shape or size i.e. it's the ability of a material to resist being bent. Examples of such materials are metals, a piece of chalk, etc.

Ductility is the ability of a material to be hammered, bent or rolled into different shapes. It therefore follows that a ductile material stretches elastically first, then stretches plastically before it breaks when a tensile force acts on it. Examples of such materials are metals such as steel and iron.

Brittleness

Brittle materials are substances that bend very little and they suddenly crack without any warning when a force acts on them. They undergo plastic deformation. Brittle materials fracture at low strains close to their elastic limit. Examples of brittle materials include glass, bronze, ceramic etc.

ELASTICITY

This is the ability of a material to regain its original length, size or shape when an applied tensile force is removed.

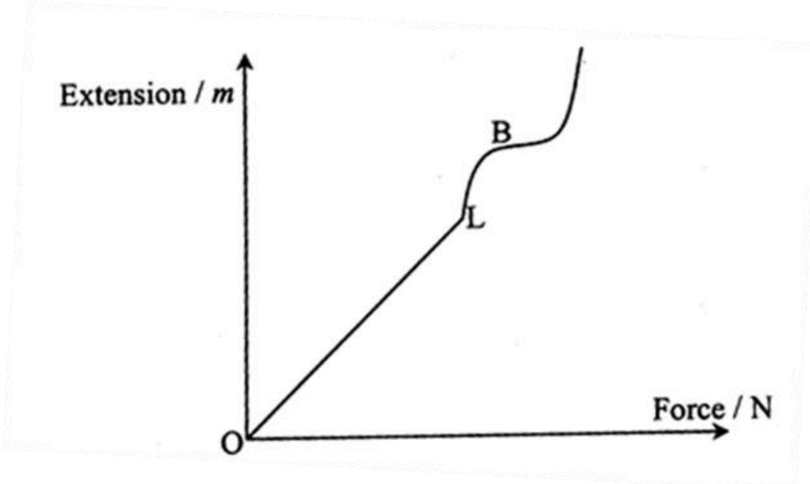
When a tensile force is applied on a material, the material stretches due to its particles being pulled further apart from one another. As a result, an extension is produced.

The extension produced depends on:

i) Size of the force applied on the material see (Hooke's law below),
ii) Nature of the material,
iii) Original length of the material (a long material produces a longer extension)
iv) Cross sectional area of the material a thin material produces a bigger extension as compared to a thick one.

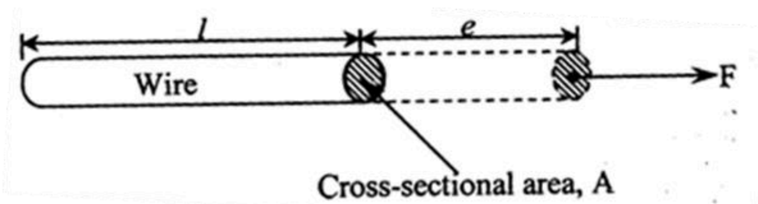
Hooke's law states that the extension of a wire is directly proportional to the applied force provided that the elastic limit is not exceeded.($F \propto e \Rightarrow F = ke$)

A graph of extension against force.



From O to L the graph is a straight line and this means that the applied force is directly proportional to the extension, and so Hooke's law is obeyed. Beyond L, the graph is no longer straight implying that the applied force is no longer directly proportional to the extension. L is called the elastic limit. When the wire stretches beyond L, it becomes plastic. Point B is called the yield point of the wire. Beyond point B, the extension of the wire increases rapidly when more load is added, and the wire finally breaks.

Tensile stress and strain



Consider a force F being applied to stretch the wire of original length l and cross sectional area, A so that the wire stretches by an eaxtension e.

Tensile stress is the force acting on an area of $1m^2$ of a material.

∴ Stress $\frac{Force}{Area}$ the SI unit of stress is Nm^2 $or\ pa.$

Dimensions of stress: [stress] $= \frac{[Force]}{[Area]} = \frac{[MLT^{-2}]}{[L^2]}$

$$= ML^{-1}T^{-2}$$

Strain is the extension per unit length.

∴ Strain = $\frac{Extension}{Original\ length}$

Strain has no units, and hence no dimensions.

Young's Modulus of elasticity (E) is the ratio of tensile stress to tensile strain.

Just like stress, the SI unit of E is Nm^{-2} or pa, and so its dimensions are $ML^{-1}T^{-2}$

Examples

1. A uniform steel wire of density $7800kgm^{-3}$ weigh 16g and is 250cm long. It lengthens by 1.2mm when stretched by a force of 80N. Calculate, the

i) Young's modulus of elasticity for steel

ii) Energy stored in the wire.

Solution

i) E = $\frac{Fl}{Ae}$ …………………… i)

Also, volume = *Al,* and still, volume = $\frac{mass}{density}$

$Al = \frac{m}{\rho} \Rightarrow A = \frac{m}{\rho l}$

Substituting for A in equation i) gives: E = $\frac{Fl}{e(^{m}/_{\rho l})} = \frac{Fl^2\rho}{em} = \frac{80\times(2.5)^2\times7800}{(1.2\times10^{-3})\times(16\times10^{-3})}$

∴ E = 2.03× $10^{11}Nm^{-2}$

ii) Energy stored = $\frac{1}{2}ke^2$, $but\ from\ hooke'slaw, F \propto e \quad \Rightarrow \quad F = ke$

Therefore, substituting for k gives: Energy stored = $\frac{1}{2}Fe = \frac{1}{2}\times 80 \times (1.2\times10^{-3})$

= $4.8\times 10^{-2}\ J$

2. A vertical wire 350cm long, diameter 0.1cm has a load of 8.50kg applied at its lower end If its Young's modulus is pa, calculate the;

i) Extension of the wire,

ii) Energy stored in the wire.

Solution

i) E = $\frac{Fl}{Ae}$

Area, A = πr^2 $\Rightarrow \frac{Fl}{\pi r^2 E} = \frac{(8.5\times9.81)\times3.5}{(2\times10^{11})\times\frac{22}{7}\times(5\times10^{-4})^2}$

Extension e = 1.9× $10^{-3}m$.

ii) Energy stored = $\frac{1}{2}ke^2 = \frac{1}{2}Fe$

$= \frac{1}{2}\times(8.5\times9.81)\times(1.89\times10^{-3}) = 8\times10^{-2}J$

Trial questions

1. The table below is for a uniform metal bar of length 63cm, width 10.8cm, and thickness 3.50mm fixed at one end and a tensile force applied at the other end.

Tension/N	170	285	505	780	815	815	815
Extension/mm	0.2	0.3	0.5	0.8	1.0	1.2	1.5

Plot a suitable graph, and from it determine Young's modulus of elasticity for the metal bar, and its yield point.

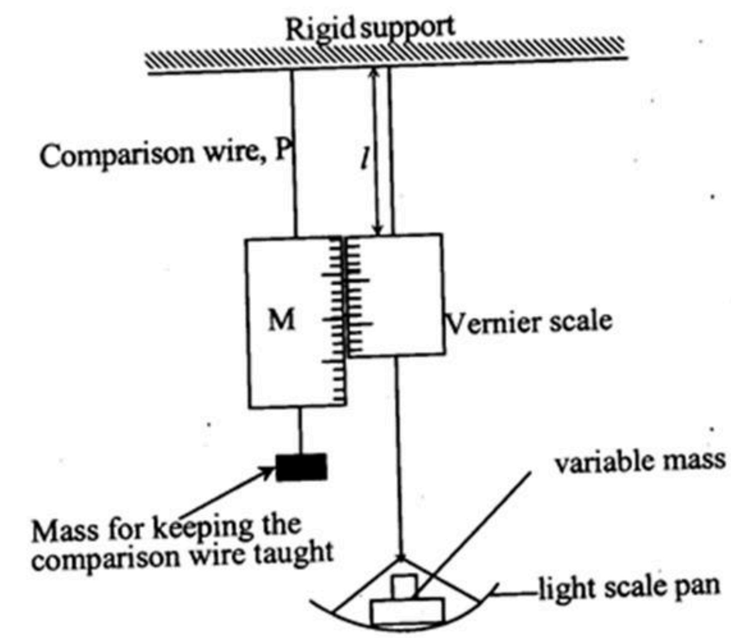
Which point (set of values) in the table would require experimental checking?

2. A copper wire 20cm long and of diameter 0.96mm stretches when loaded according to the table below.

Load/kg	0	2	4	6	8	10
Extension/mm	0	0.7	1.4	2.0	2.8	3.45

Plot a suitable graph, and from it determine Young's modulus of elasticity for the copper wire in question. Also find the work done in stretching the wire when a load of 9kg is applied.

Experimental determination of Young's modulus of Elasticity of a ductile material



Two thin, long wires P and Q of the same material are suspended from a common rigid support.

Initial loads are added to the ends of P and Q to remove kinks i.e. to make them taut.

The diameter of wire Q is measured by a micrometer screw gauge at several places and the average, d value calculated. The cross sectional area $A = \pi\frac{d^2}{4}$ is determined.

The original length, l of the wire is measured. Various loads are added at the end of the test wire Q and the corresponding extensions caused are read from the vernier scale, but after each reading, the load should be removed to check that the wire returns to its original position, implying that the elastic limit has not been exceeded.

The original length, l of the test wire is measured from the rigid support to the vernier scale.

The results are tabulated and a graph of load against extension is plotted, and its slope, s determined

Young's modulus, $E = \frac{sl}{A}$

Alternatively A graph mass (in kg) is plotted against extension in meters, then,

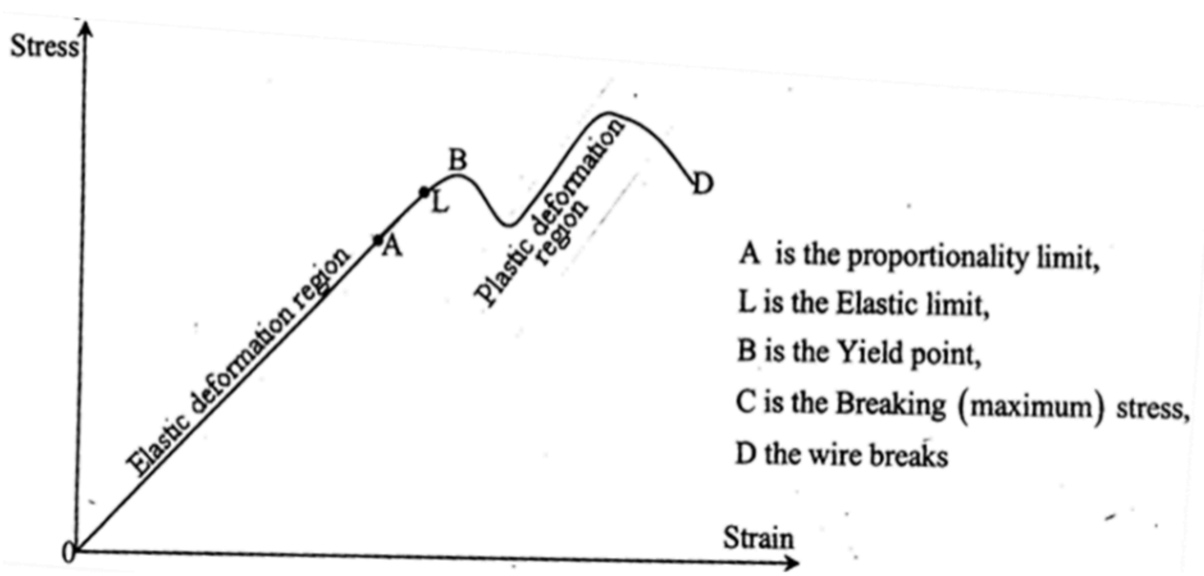
Young's modulus, $E = \frac{slg}{A}$

Note: the test wire should be long such that measurable extensions are produced. It should also be thin such that there is no need of use of bigger stretching masses in case the wire is thick which are difficult to handle, especially in the lab.

Two identical wires are used to eliminate errors that would result due to any changes in position due to bending of the rigid support (yielding of the support when loads are added to Q). It also eliminates errors due to changes in temperature.

It is left to the student to establish why the wire should be free of kinks at first.

A graph of stress against strain or a ductile material



Between O and L, the material returns to its original length when the applied stress is reduced to zero. If the material is stretched beyond the elastic limit, the molecules of the material slide over each other and the material becomes plastic. On reducing the stress to zero, the material does not recover its original length. As the long is increased further, the extension increases rapidly and the wire becomes narrower and finally breaks.

Therefore, the breaking stress of a material is the corresponding force per unit area of the narrowest cross section of the material (wire)

Elastic limit is the maximum load which a body can experience and still regain its original size and shape when the load is removed.

OR

It's the load beyond which the material stops undergoing elastic deformation and starts undergoing plastic deformation.

Yield point is the load beyond which the material stops undergoing plastic deformation and after any further increase in load leads to a rapid increase in extension until the material breaks.

Elastic deformation region is the region over which a material returns to its original length when the stress is removed.

Plastic deformation region is the region over which a material does not return to its original length when the stress is removed. The material retains some of the extension.

It is instructive to note that during plastic deformation, the mechanical energy gained by the stretched wire is dissipated as heat, and therefore, unlike elastic deformation, the energy is not recovered when the load is removed.

Energy stored in a stretched wire
Consider a wire of original length *l* stretched by and amount e on applying a force, F, such that the elastic limit is not exceeded.
When the wire is unstretched, the force acting on it is zero.

Therefore, the average stretching force = $\frac{0+F}{2} = \frac{1}{2}F$

Work done in stretching the wire = (average force)× (distance) = $(\frac{1}{2}F) \times (e)$

But the work done in stretching the wire is stored as energy in the wire

∴ Energy stored in a stretched wire = $\frac{1}{2}Fe$

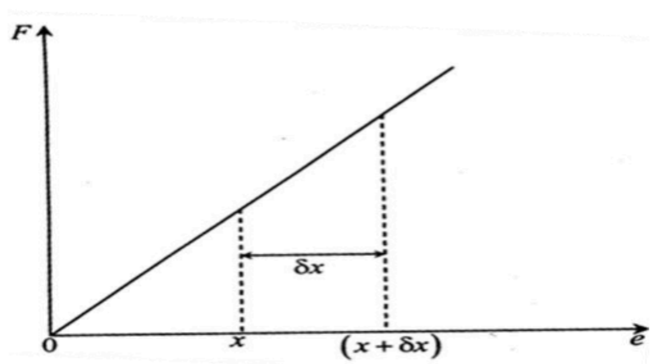
From E = $\frac{Fl}{Ae}$, $F = \frac{EeA}{l}$

Substituting for F in the expression for energy gives:

$$\text{Energy stored} = \frac{1}{2} \times \left(\frac{EeA}{l}\right) \times e$$

$$= \frac{Ee^2A}{2l} \quad \text{........(i)}$$

Alternatively: consider the graph of force against extension as shown below



The small work done δw in extending the wire from x through a small distance to $(x+\delta x)$ is given by: $\delta w = F\ \delta x$.

But from Hooke's law, $F \propto x \Rightarrow F = kx$

$\delta w = kx\ \delta x \Rightarrow dw = kxdx$

The total work done in increasing the extension from 0 to e is therefore given by:

$$w = \int_0^e kxdx = \left.\frac{kx^2}{2}\right]_0^e$$

Therefore, work done in stretching the wire $= \frac{1}{2}\ ke =$ Energy stored by the wire.

It can be noted that this work done/ energy stored is equal to the area under the graph of force against extension.

Comparing this energy stored with the one derived earlier, i.e. equation shows:

Energy stored $= \frac{1}{2}\ ke^2$ and that energy stored $= \frac{Ee^2A}{2l}$

Equating the two 'energies' gives: $\frac{1}{2}\ ke^2 = \frac{Ee^2A}{2l}$

$$\Rightarrow k = \frac{EA}{l}$$

Energy stored per unit volume

For a wire of cross sectional area A, and original length *l*.

Volume $= Al$ and since energy $= \frac{1}{2}\ Fe$

Energy stored per unit volume $= \frac{Energy\ stored}{Volume} = \frac{(^1/_2 Fe)}{Al} = \frac{Fe}{2Al} = \frac{1}{2} \times \left(\frac{F}{A}\right) \times \left(\frac{e}{l}\right)$

But $\frac{F}{A} = stress, and\ \frac{e}{l} = strain.$

$\therefore$ Energy stored per unit volume $= \frac{1}{2} \times (stress) \times (strain)$

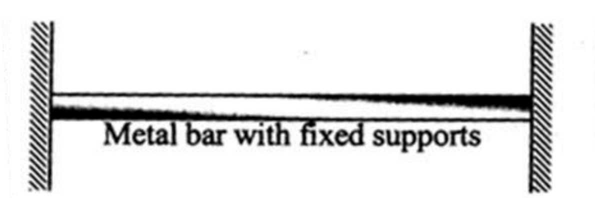
<u>Trial question</u>

A wire of length *l* and cross- sectional area, A has a force constant k. the wire is stretched to a length $(l + x)$ by a constant force.

i) Assuming Hooke's law, show that $k = \frac{EA}{l}$

ii) Show that the energy stored per unit volume of the wire $= \frac{1}{2}E\ (\frac{x}{l})^2$

Force in a metal bar due to expansion



If a metal bar is heated, it will expand. However, if the bar is prevented from expanding by fixed supports, then large forces will be set up with in the metal bar. These forces will be equal to the tensile force that would have caused the same extension if the bar was free to expand

The coefficient of linear expansivity, $\alpha = \frac{change\ in\ length}{(original\ length)\times(temperature\ change)}$

$$= \frac{e}{l\Delta\theta} \Rightarrow e = \alpha/(\Delta\theta)$$

But $F = \frac{EeA}{l} \Rightarrow F = \frac{E(\alpha l(\Delta\theta))A}{l} = E\alpha(\Delta\theta)A$

Energy stored $= \frac{1}{2}Fe = \frac{1}{2}\times(E\alpha(\Delta\theta)A)\times e^2 = \frac{1}{2}E\alpha^2(\Delta\theta)^2 Al$

Examples

1. A uniform metal bar of length 1.0 cm and 2cm diameter is fixed between two rigid supports at 25^0C. Young's modulus for the metal bar is 2.0×10^{11} pa, and the coefficient of linear expansivity, $\alpha = 1.0\times10^{-5}k^{-1}$. If the temperature of the rod is raised to 75^0C,
2. Find the :
   i) Force exerted on the supports,
   ii) Energy stored in the rod at 75^0C.

Solution

i) $F = \frac{EeA}{l} \Rightarrow F = \frac{E(\alpha l(\Delta\theta))A}{l} = E\alpha(\Delta\theta)A$

But area A $= \pi r^2 = \frac{22}{7}\times(0.01)^2 = 3.14\times10^4 m^2$

F $= (2.0\times10^{11})\times(1.0\times10^{-5})\times(50)\times(3.14\times10^{-4}) = 31400N$

ii) Energy stored $= \frac{1}{2}F\alpha l(\Delta\theta) = \frac{1}{2}\times(31400)\times(1.0\times10^{-5})\times(1)\times(50) = 7.85$J

2. A metallic rod of temperature has its ends rigidly fixed. If the temperature of the rod is reduced to θ_0 ^{0}C find the stress in the rod in terms of young's modulus, E, the coefficient of linear expansivity, α and the temperature change.

Solution

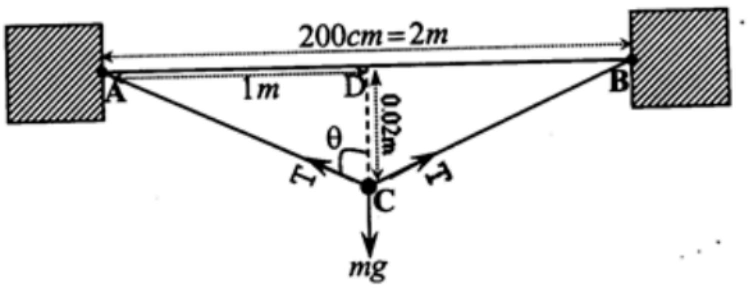
$E = \frac{Stress}{Strain} \Rightarrow Stress = (E)\times(Strain) = E\frac{e}{l}$

But e $= \alpha l\Delta\theta = \alpha l(\theta - \theta_0)$

$\Rightarrow$ Stress $= E\times\left[\frac{\alpha l(\theta-\theta_0)}{l}\right] = E\alpha(\theta-\theta_0)$

3. A copper wire of length 200cm and diameter 1.22mm is fixed horizontally to two rigid supports which are 200cm apart. Find the mass of the load which when suspended at the midpoint of the wire produces a sag (depression) of 2cm at that point.

Solution



$$(AC)^2 = (1)^2 + (0.02)^2 \implies AC = 1.0004m \quad \therefore BC = 1.0002m$$

Therefore new length of the wire = AC + BC = 1.0002 + 1.0002 = 2.0004m

$\implies Extension = 2.0004 - 2.0000 = 0.0004m$

Area of wire = πr^2

r = 0.61mm = $6.1\times 10^{-4}m \implies A = \frac{22}{7} \times (6.1 \times 10^{-4}) = 1.17 \times 10^{-6}m^2.$

From F = $\frac{EeA}{L}$, *the force is the tension in the wire*, $\implies T = \frac{EeA}{l}$

$\frac{(12.3\times10^{10})\times(1.17\times10^{-6})\times(4\times10^{-4})}{2}$ = 28.8N

Resolving vertically; (↑): $2Tcos\theta = mg \implies m = \frac{2Tcos\theta}{g}$ *where* $\cos\theta = \frac{DC}{AC}$

$= \frac{2\times(28.8)\times(^{0.02}/_{1.0002})}{9.91}$ = 0.1174kg

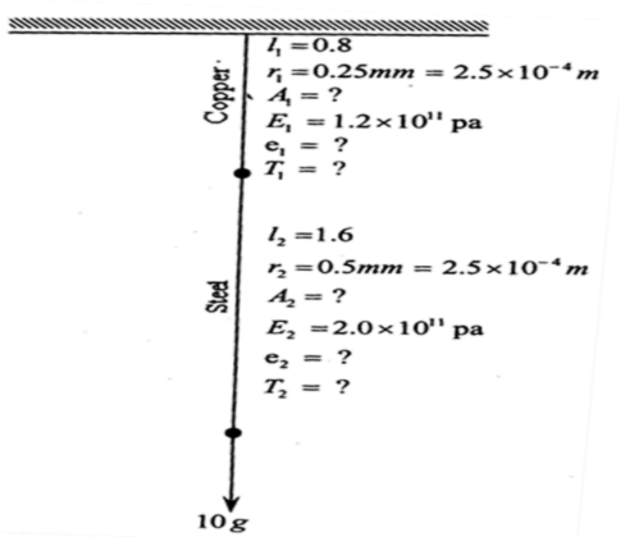
= 117.4g

4. One end of a copper wire of length 0.8cm and diameter 0.5mm is welded to a steel wire of length 1.6m and diameter 1.0mm, while its other end is fixed. A load of 10kg is suspended from the free end of the steel wire. If young's modulus of steel E_c is 1.2×10^{11} pa, calculate the;

i) Extension which results,

ii) Energy stored in the compound wire.

Solution



$$A_1 = \pi {r_1}^2 = \frac{22}{7} \times (2.5 \times 10^{-4})^2 = 1.96 \times 10^{-7} m^2$$
$$A_2 = \pi {r_2}^2 = \frac{22}{7} \times (5 \times 10^{-4})^2 = 7.85 \times 10^{-7} m^2$$

It can be recalled that if the wires are in series as in question, then the stretching force is the same for both wires. However, if the wires are arranged in parallel as will be seen in latter examples, then it's the extension that is the same.

From $F = \frac{EeA}{l}$, $\quad \therefore T_1 = \frac{E_1 e_1 A_1}{l_1}$

$$\Rightarrow 10 \times 9.81 = \frac{(1.2 \times 10^{11}) \times (1.96 \times 10^{-7}) \times e_2}{0.8}$$

$\therefore e_2 = 1 \times 10^{-3} m$

Therefore the total extension $e_1 + e_2 = 4.44 \times 10^{-3} m$

Energy stored $= \frac{1}{2} Fe = \frac{1}{2} \times (10 \times 9.81) \times (4.44 \times 10^{-3}) = 0.213$ Joules

5. A uniform wire of unstretched length is attached to two points A and B which are 2.4m apart and in the same horizontal line. When a 6kg mass is attached to the midpoint, C of the wire, the equilibrium position of C is 0.52m below the line AB. Neglecting the weight of the wire and taking its modulus of elasticity to be $2 \times 10^{11} Nm^{-1}$, find the:

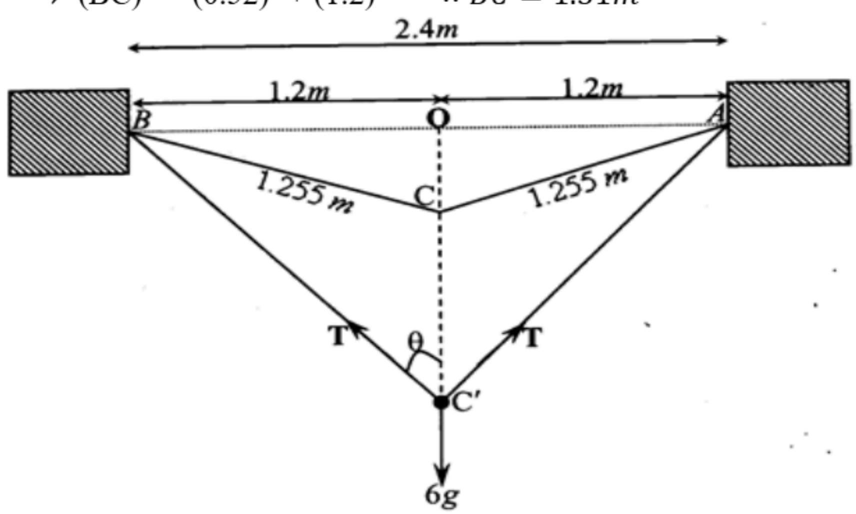
i) Strain in the wire,
ii) Stress in the wire,
iii) The energy stored in the wire,
iv) State any assumptions made.

Solution

Consider triangle BOC:

$(BC)^2 = (OC)^2 + (OB)^2$

$\Rightarrow (BC)^2 = (0.52)^2 + (1.2)^2 \quad \therefore BC = 1.31m$



$\therefore \; 2 \times (BC) = \; 2 \times 1.31 \; = 2.62m$

Original length = 2.51 $\Longrightarrow extension = 2.62 - 2.51 = 0.11m$

Strain = $\frac{e}{l} = \frac{0.11}{2.51} = 0.044$

(ii) Stress = $E \times Strain = (2 \times 10^{11}) \times (0.044) = 8.8 \times 10^{9} pa$

(iii) $\cos\theta = \frac{0.52}{1.31} \Rightarrow \theta = 66.6^{0}$

Resolving vertically at point C' *gives:*

$T\cos\theta + T\cos\theta = 6\text{g}$

$\Rightarrow T = \frac{6\text{g}}{2\cos\theta} = \frac{6\times9.81}{2\times\frac{0.52}{1.31}} = 74.13\ N$

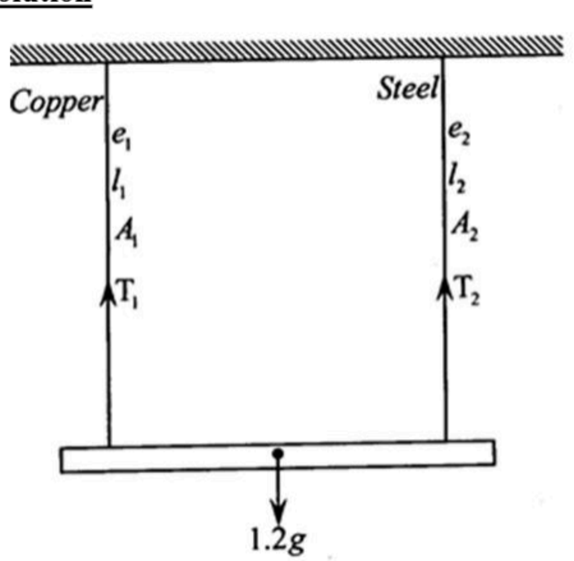
Energy stored $= \frac{1}{2}Fe = \frac{1}{2} \times T \times e$

$= \frac{1}{2} \times 74.14 \times 0.11 = 4.077$ Joules

(iv) It is assumed that the elastic limit is not exceeded.

6. A copper wire and steel wire each 2.0cm long and 3.0mm^2 cross-sectional area are laid side by side on a rigid support. The composite wire is placed vertically with the lower end supporting a mass of 1.2kg. Assuming that E for steel is 2.0$\times 10^{11}$ pa and that for copper is 1.2 $\times 10^{11}$ pa, calculate the tension in each wire, the potential energy of the whole system, and state any assumptions made

Solution



From $E = \frac{Fl}{eA}$, $F = \frac{EeA}{l}$

$T_1 = \frac{E_1e_1A_1}{l_1}$ and $T_2 = \frac{E_2e_2A_2}{l_2}$

But $T_1 + T_2 = 1.2\text{g}$

$\Rightarrow \frac{E_1e_1A_1}{l_1} + \frac{E_2e_2A_2}{l_2} = 1.2\text{g}$

From the question, $A_1 = A_2 = 3 \times 10^{-6}m^2$, $l_1 = l_2 = 2.0m$

And since the wires are in parallel, then the extensions produced in either wire are the same

$\Rightarrow e_1 = e_2 = e$

$\therefore (E_1 + E_2)\frac{Ae}{l} = 1.2\text{g}$

$$\Rightarrow (2.0 \times 10^{11} + 1.2 \times 10^{11})\frac{(3\times10^{-6})}{1}e = (1.2 \times 9.81)$$

$\therefore$ e = $1.22625\times 10^{-5}m$

$$\therefore\ T_1 = \frac{E_1e_1A_1}{l_1} = \frac{(1.2\times10^{11})\times(1.22625\times10^{-5})\times(3\times10^{-6})}{1} = 4.4145\text{N} \quad \text{and}$$

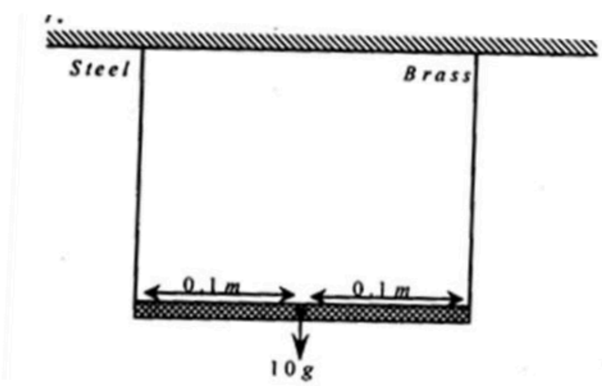
$$\therefore\ T_2 = \frac{E_2e_2A_2}{l_2} = \frac{(2.0\times10^{11})\times(1.22625\times10^{-5})\times(3\times10^{-6})}{1} = 7.3575\text{N}$$

$$\text{Potential energy} = \frac{1}{2}Fe = \frac{1}{2}\times(T_1 + T_2)\times e$$

$$= \frac{1}{2}\times(4.4145 + 7.3575)\times(2.45\times10^{-5}) = 14.44\times10^{-5}\ \text{Joules}$$

The assumption made is that the elastic limit is not exceeded

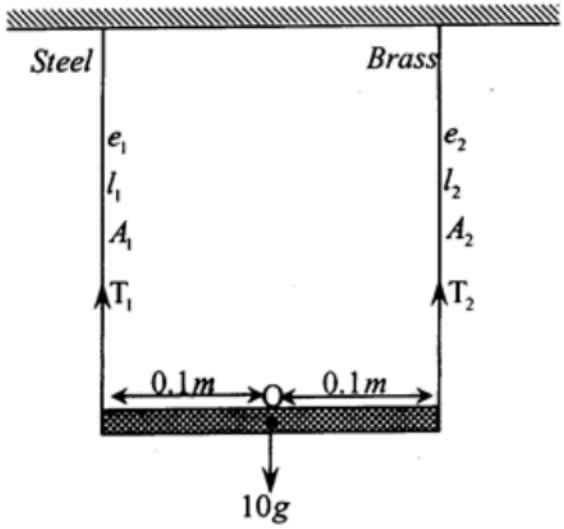
7. A light rigid bar is suspended horizontally from two vertical wires one made of steel and the other made of brass, as shown in the figure.



Each wire is 2m long. The diameter of the steel wire is 0.6mm, and the length of the light rigid bar is 0.2m. When a mass of 10kg is suspended from the centre of the light rod, the bar remains horizontal. If young's modulus of elasticity for steel, E_s =2.0×10^{11}pa, and that of brass E_b =1.0×10^{11} pa, calculate the

i) Tension in each wire ,
ii) Extension of the steel wire and the energy stored in it,
iii) Diameter of the brass wire
iv) If the brass wire were replaced by another brass wire of diameter 1mm, where would the mass be suspended so that the light rod remains horizontal?

Solution



For steel: For brass:

$d_1 = 0.6mm \Rightarrow r = 0.3mm = 3 \times 10^{-4}$ $d_2 =?$

For equilibrium of the light rigid rid,

$T_1 + T_2 = 10\text{g} \ldots\ldots\ldots\ldots..\text{(i)}$

Taking moments about point O gives; $0.1 \times T_1 = 0.1 \times T_2 \Rightarrow T_1 = T_2$

Substituting for T in equation i) gives: $2T_1 = 10\text{g} \Rightarrow T_1 = T_2 = 5\text{g}$

(ii) From $F = \frac{EeA}{l}$, $T_1 = \frac{E_s e_1 A_1}{l_1}$

$$e_1 = \frac{T_1 l_1}{E_s A_1} = \frac{(5\times 9.81)\times 2}{(2\times 10^{11})\times(2.83\times 10^{-7})} = 0.0174$$

$$e_1 = e_2 \Rightarrow e_2 = 0.0174$$

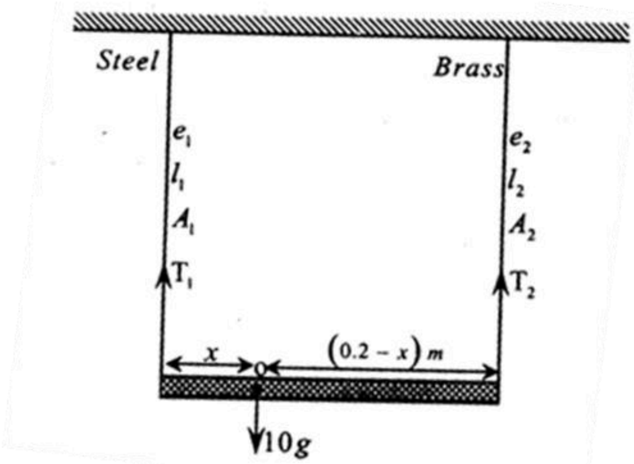
Energy scored $= \frac{1}{2}Fe = \frac{1}{2}\times(5\times 9.81)\times(0.0174) = 4.27\times 10^{-1}$Joules

(iii) $T_2 = \frac{E_b e_2 A_2}{l_2} \Rightarrow A_2 = \frac{T_2 l_2}{E_b e_2} = \frac{(5\times 9.81)\times 2}{(1\times 10^{11})\times(0.0174)} = 5.6\times 10^{-8}m^2$

But area $A_2 = \pi r^2 \Rightarrow r = \sqrt{\frac{A_2}{\pi}} = \sqrt{\frac{5.6\times 10^{-8}}{\frac{22}{7}}} = 1.34\times 10^{-4}m$

$\therefore d_2 = 2r = 2.68\times 10^{-4}m$

(iv)



For this case, diameter for brass, d = 1mm $\Rightarrow r = 5 \times 10^{-5}m$

$$\Rightarrow A_2 = \pi r^2 = \frac{22}{7} \times (5 \times 10^{-5})^2 = 7.85 \times 10^{-7}m^2$$

Taking moments about point O gives: $xT_1 = (0.2 - x)T_2$

But, $T_1 = \frac{E_s e_1 A_1}{l_1}$ and $T_2 = \frac{E_b e_2 A_2}{l_2}$

$$\Rightarrow \left(\frac{E_s e_1 A_1}{l_1}\right)x = \left(\frac{E_b e_2 A_2}{l_2}\right)(0.2 - x)$$

Also $e_1 = e_2$ and $l_1 = l_2 \Rightarrow E_s A_1 x = E_b A_2 (0.2 - x)$

$$(2 \times 10^{11}) \times (2.82 \times 10^{-7}) \times (x) = (1 \times 10^{11}) \times (7.85 \times 10^{-7}) \times (0.2 - x)$$

$$\Rightarrow 56400x = 78500(0.2 - x) \quad \therefore x = 0.116$$

Therefore, the mass should be hanged at 0.116m from the steel wire.

8. A rubber cord of a catapault has a cross sectional area of 1.2mm^{-2} and original length and 0.72m, and is stretched to 0.84m to fire a small stone of mass 15g at a bird. Calculate the initial velocity of the stone when it just leaves the catapault. Assume that young's modulus for rubber is 6.2$\times$ 10^{8}Nm^{-2}. State any assumptions made.

Solution

Force stretching rubber, F $= \frac{EAe}{l} = \frac{(6.2\times10^8)\times(1.2\times10^{-6})\times0.12}{0.72}$ = 124N

Energy stored in rubber $= \frac{1}{2}Fe = \frac{1}{2} \times 124 \times 0.12 = 7.44$J

Kinetic energy of stone $= \frac{1}{2}mv^2$

$\therefore \frac{1}{2}mv^2 = 7.44 \Rightarrow \frac{1}{2} \times 0.015 \times v^2 = 7.44$

$\Rightarrow v = 31.5ms^{-1}$

Trial questions

1. A copper wire and steel wire each of length1.5m and diameter 2mm are joined end to end to form a composite wire. The composite wire is loaded until its length becomes 3.003m. If young's modulus of steel is 2.0$\times$ 10^{11} pa and that hat of copper is 1.2 $\times$ 10^{11}pa,
   i) Find the strains in the copper and steel wire,

ii) Calculate the force applied.

[Ans. Strain in copper = 0.0013, strain in steel = 7.5×10^{-4}. Force = 4.7×10^{2}N]

2. Two identical cylindrical steel bars each of radius 3.0mm and length 7m rest in a vertical opposition with their lower ends on a rigid horizontal surface. A mass of 4.0kg is placed on the top of one bar. The temperature of the other bar is to be altered so that the two bars are once again of equal length. Given that the coefficient of linear expansivity of steel is $1.2 \times 10^{-5} K^{-1}$

(i) By how much should the temperature be altered?

(ii) Find the energy stored in the bar due to the temperature change

[Ans: 0.58K, 9.6×10^{-1}J]

3. Two wires of steel and phosphor bronze each of diameter 0.40cm and length 3.0m are joined end to end to form a composite wire of length 6.0m calculate the tension in the wire needed to produce a total extension of 0.128cm in the composite wire.

(Given: $E_s = 2.0 \times 10^{11} Nm^{-2}$ and $E_b = 1.2 \times 10^{11} Nm^{-2}$)

[Ans: 100.5N]

Work hardening

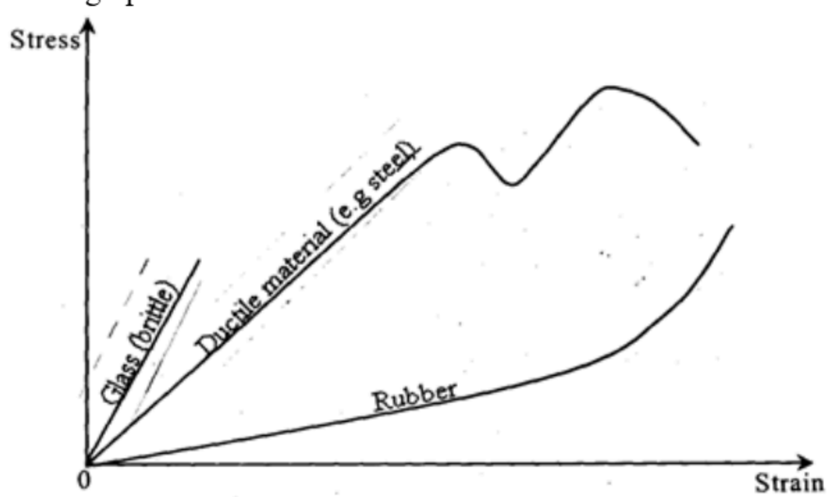
When a metal is deformed by bending it repeatedly, it becomes harder and brittle, and its resistance to plastic deformation increases. The increase in deformation leads to an increase in the density of dislocations. This process is called work hardening of metals.

Plastic deformation

Metals have certain planes which are rich in atoms l these planes are called atomic or crystal planes. During plastic deformation, some atomic planes of the material slide over each other. Movement of dislocations takes place, and on removing the stress, the material does not recover its original length and shape. There is also a corresponding loss in mechanical energy.

Elasticity of non – metals

Consider this graph.



Glass has only a small elastic region, no plastic region (deformation), and fractures easily.

Its brittle and concentration of stress at any crack in glass makes it break. It also has no yield point

Rubber stretches very easily without breaking compared to metals such as steel and copper. It's less stiff than metals. The range of elasticity for rubber is greater, and hence its value of Young's modulus of elasticity is much smaller than that of most metals.
Such behavior for rubber is due to its molecular structure. Rubber consists of coiled molecules and the molecules uncoil when it's stretched. It becomes stiff when the molecules are fully stretched. Rubber does not undergo plastic deformation.

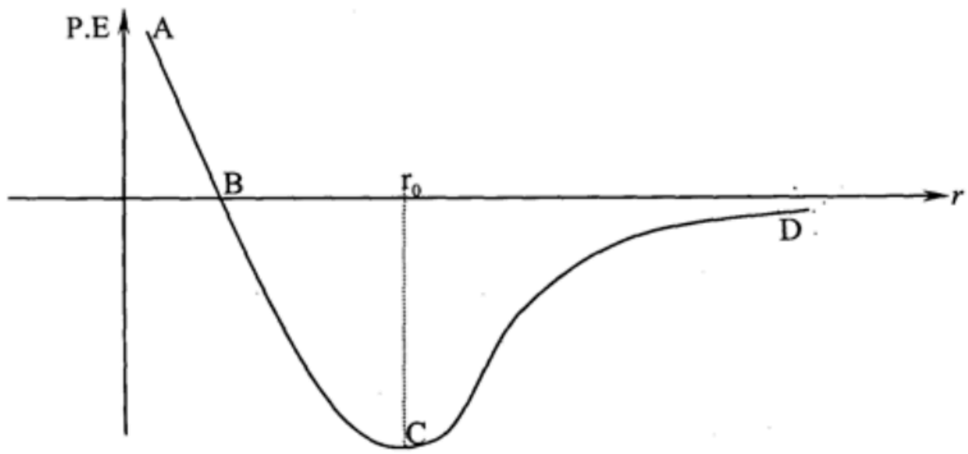
Potential energy and force between molecules

The potential energy of the molecules is due to the interactions between given molecules and the surrounding molecules.

P.E between molecules is given by; $V = \frac{a}{r^p} - \frac{b}{r^q}$

Where *a and b* are constants p and q are powers of molecular separation, r the positive term with constant a indicates a repulsive force and the negative term with constant b an attractive force.

Graph of potential energy (V) against molecular separation, r

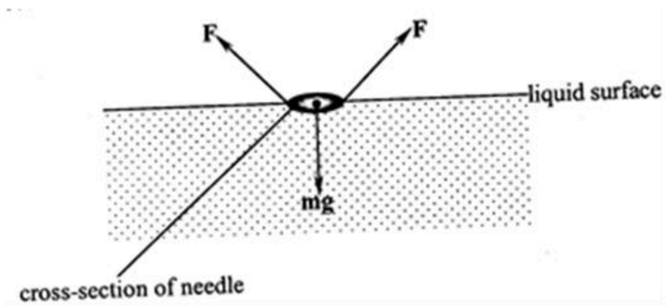


A long part AB the potential energy is positive and P.E is negative along part BCD. For a distance $r < r_0$, repulsive forces exceed attractive forces between the molecules. At separation $r = r_0$, the attractive forces and repulsive forces balance.. The potential energy is therefore minimum at $r = r_0$ which corresponds to equilibrium separation of molecules, at absolute zero, where the thermal kinetic energy is zero.
F or $r > r_0$ attractive forces exceed repulsive forces so that molecules return to equilibrium position when slightly displaced from it. So the molecules of a solid oscillate about their equilibrium or mean position.

CHAPTER 9: SURFACE TENSION

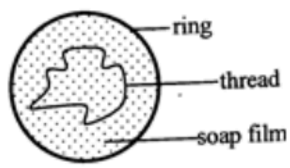
Common observations that are explained by surface tension

1. A drop of water may remain clinging on to the tap as if the water particles were held in a bag.
2. Some insects such as pond skaters are able to walk over the surface of the water without getting wet.
3. A steel needle can be made to float on water despite its greater density.

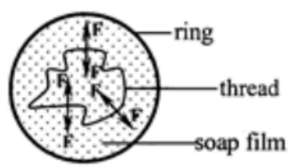


The floating needle creates a depression in the liquid surface so that the surface tensional forces, F which act in the surface now have upward directed components which are capable of supporting the weight of the needle.

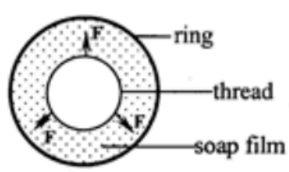
4. Consider a thread placed on a sop film which is supported by a ring as shown below.



There are equal and opposite forces (surface tensional forces) on each side of the thread and therefore the thread stays where it has been placed.



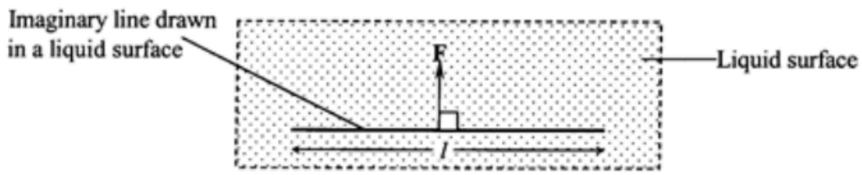
However if the soap film is punctured or broken in the middle of the loop in the region bounded by the thread, there are no more forces inside the thread, and so the only forces acting on it are outward.



As a result, the thread is pulled into a circle.

Therefore, surface tension is a property which makes a free liquid surface act like a stretched skin. Surface tension is due to intermolecular attraction in the liquid surface and it's these forces that produce a skin effect on the surface.

Consider a liquid surface below



Definition: Surface tension (γ) is the force acting perpendicular to one side of an imaginary line of length one meter drawn in the surface of a liquid

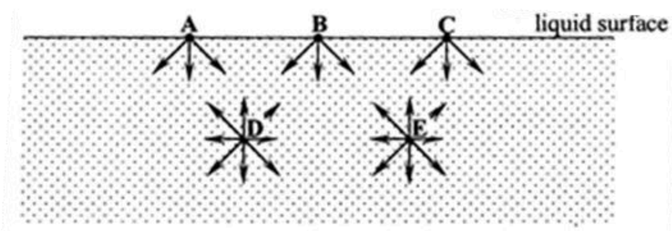
The S I unit of γ *is* Nm^{-1}.

Dimensions of γ: $[\gamma] = \frac{[Force]}{[Length]} = \frac{MLT^{-2}}{L}$

$$\therefore [\,\gamma\,] = MT^{-2}$$

Molecular theory explanation of surface tension

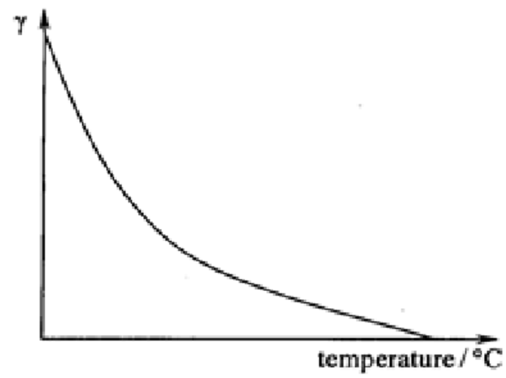
Consider a liquid shown below



Molecules such as D and E deep inside the liquid are attracted equally in all directions by the surrounding molecules. The attractive forces balance the repulsive forces and therefore, the average intermolecular forces between them and the surrounding molecules is zero.

However for molecules A, B and C in the liquid surface, the average molecular separation is greater than that of the molecules deep inside. Since there are more liquid molecules below them than above, molecules in the liquid surface experience greater attraction from their neighbours below. This puts the molecules in the liquid surface in a state of tension and hence the liquid surface acts like an elastic skin. This phenomenon is called surface tension.

Effect of temperature on surface tension



When the temperature of a liquid increases, the mean kinetic energy of the molecules of the liquid increases. The forces of attraction between the molecules will decrease since the molecules spend less time in the neighborhood of a given molecule; hence surface tension decreases with increase in temperature as shown in the graph.

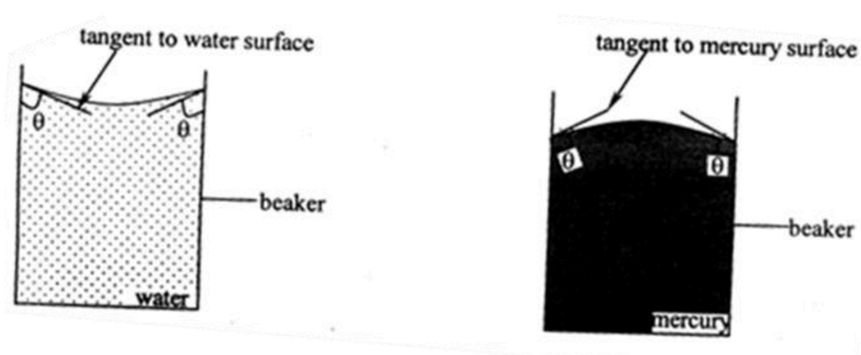
Experiment to show that surface tension of a liquid decreases with increase in temperature

Procedure
Lycopodium powder or some light dust is sprinkled on the surface of water in a flat metal dish. One side of the dish is heated with a candle.

Observation
It will be noted that the particles of the powder will be swept away from the heated portion, implying that the surface tensional forces can longer hold the particles in their previous positions.

Conclusion
Surface tension of liquids decreases with increase in temperature.
It should however be noted that impurities, detergents or soap solution also decrease the surface tension of a liquid.

Angle of contact



When water is poured in a clean beaker or a capillary tube, the meniscus curves downwards, whereas for mercury the meniscus curves upwards.

Angle of contact is the angle between the solid surface and the tangent to the meniscus at the point where it touches the liquid, and measured through the liquid.

For water, θ is acute, i.e. less than 90°. This is due to the fact that the adhesion forces between the liquid and the solid surface is greater than the cohesion forces between the liquid molecules.

For mercury, θ is obtuse, i.e. $90^0 < \theta < 180°$. This due to the fact that the cohesion forces between the liquid molecules is greater than the adhesion forces between the liquid molecules and the solid surface.

Factors affecting magnitude of angle of contact

1. Nature of the liquid,
2. Nature of the surface of the container (solid surface),
3. Impurities of the liquid.

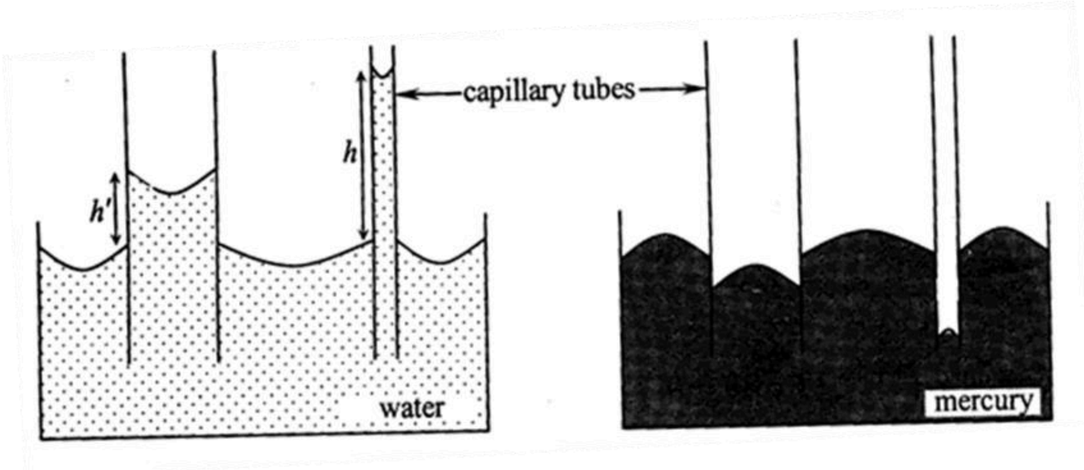
The reader should note that liquids with acute angles of contact such as water rise in a capillary tube, and for the same reason, they spread over and wet a clean glass surface when split on it. On the other hand liquids with obtuse angles of contact such as mercury are depressed in a capillary tube. For the same reason, mercury gathers itself into pools and forms drops when spilt on a clean glass surface. It therefore does not wet the glass.

Experiment to determine angle of contact

A clean flat plate is dipped into the liquid and tilted until the liquid surface on one side of the plate is horizontal up to the line of contact.
The angle α between the flat plate and the liquid surface is measured by means of a protractor, suitably placed against the edge of the plate. The angle of contact $\theta = 180 - \alpha$

CAPILLARITY
When a capillary tube is immersed in a beaker with water, the water rises in the tube to a height above the surface due to surface tension. The narrower the tube, the greater the height. This is due to the fact that the adhesive forces between water molecules and glass molecules are greater than the cohesive forces between the water molecules. The water therefore rises up the tube so that more water molecules are in contact with the glass, and a concave meniscus is formed. When on the other hand the capillary tube is placed inside mercury, the liquid is depressed below the outside mercury level. The depression decreases as the diameter of the capillary tube increases.

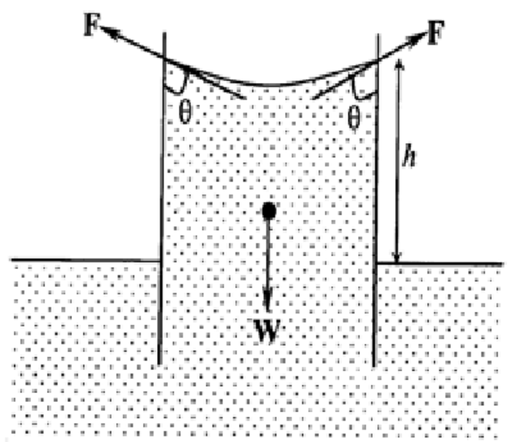
This is because the cohesive forces between mercury molecules are greater than the adhesive forces between the mercury and the glass molecules. Mercury therefore sinks down the tube such that more mercury molecules remain together, and a convex meniscus is formed.



Definition: Capillarity is the rise or fall of a liquid in a capillary tube.

Tube of insufficient length
If a capillary tube of insufficient length is put in a liquid, the liquid rises to the top of the tube the meniscus changes its shape until an equilibrium at a smaller height is reached, and the meniscus acquires a new radius of curvature.

Derivation of an expression for h
Consider a liquid which rises up in a clean glass capillary tube, the liquid stops rising when the weight of the raised column acting vertically downwards equals the vertical component of the upward forces exerted by the tube on the liquid.



Resolving vertically, $(\uparrow), F cos\theta = W = mg \quad \Rightarrow \quad F cos\theta = mg$

But mass, m = (volume)× $(density) = v\rho, and\ volume, v = \pi r^2 h$

$$F\ cos\theta = \pi r^2 h\rho g \quad \Rightarrow \quad h = \frac{Force}{\pi r^2 \rho g} \ldots\ldots\ldots\ldots\ldots i)$$

Substituting for Fin equation i) gives: $h\ = \frac{(2\pi r\gamma)cos\theta}{\pi r^2\rho g}$

$$\therefore\ h = \frac{(2\pi r\gamma)cos\theta}{\pi r^2\rho g}$$

Note: if θ is acute, then cos θ is positive and therefore h is positive, implying that the liquid rises up in the tube. If θ is obtuse, $cos\theta$ is negative and therefore h is negative, implying that the liquid falls in the capillary tube, below the level of the surrounding.

Example

A capillary tube of diameter 0.4mm is placed vertically inside:

i) A liquid of density 800kgm^{-3} and surface tension $5\times 10^{-2}Nm^{-1}$ $and\ angle\ of\ contact\ 30^0,$

ii) Mercury of angle of contact and surface tension 0.52.

For each case, calculate the height to which the liquid rises in the tube.

Solution

d = 0.4mm $\Rightarrow\ r = 0.2\times 10^{-3}m$

i) $\gamma = 5\times 10^{-2},\ \theta = 30^0$

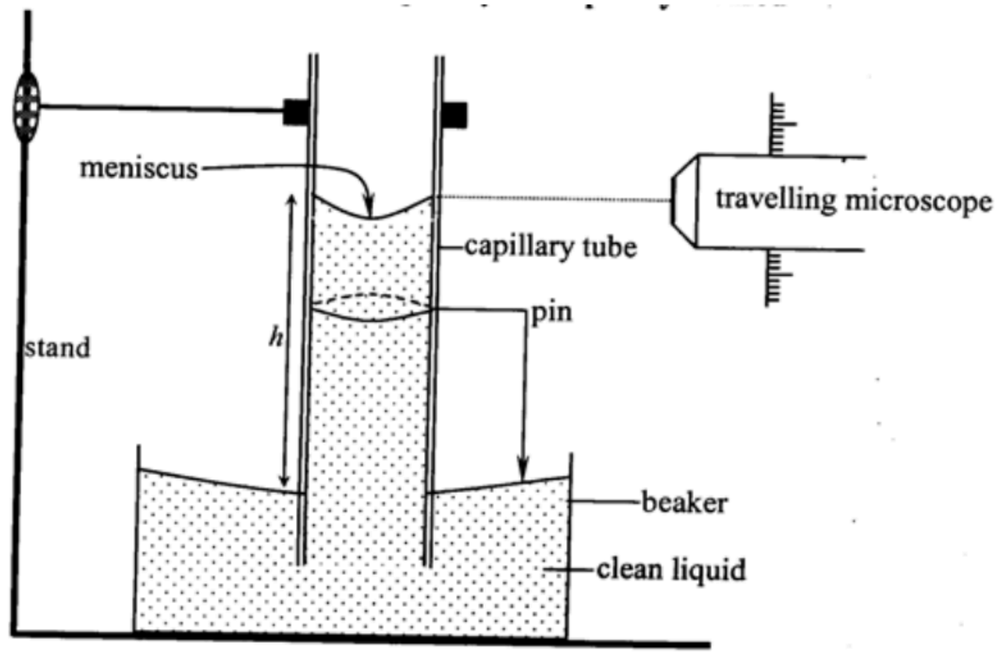
$$h = \frac{2\gamma cos\theta}{\gamma\rho g} = \frac{2\times(5\times10^{-2})\times cos30^0}{(0.2\times10^{-3})\times800\times9.81} = 0.032m$$

ii) $\gamma = 0.52,\ \theta = 139^0,\ \rho = 13600$

$$h = \frac{2\gamma cos\theta}{\gamma\rho g} = \frac{2\times(0.52)\times cos139^0}{(0.2\times10^{-3})\times13600\times9.81} = -0.0294m$$

The negative sign just means a depression.

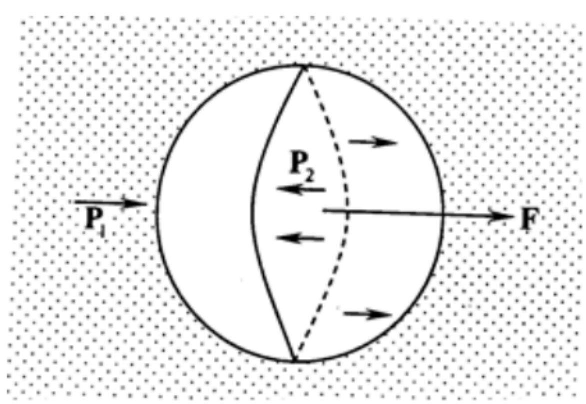
Measurement of surface tension of a liquid by the capillary method.



A clean capillary tube which is supported by a stand is placed in a beaker containing a clean liquid of known density,ρ and angle of contact, θ.
A pin bent at right angles at two paces is tied to a capillary tube with a rubber band. The pin is adjusted until its sharp point just touches the horizontal level of the liquid in the beaker. A travelling microscope is focused on the meniscus and the reading, s_1 on the scale is recorded. The beaker is removed and the microscope is focused on the tip of the pin. The scale reading s_2 is recorded. The liquid column, h = $|s_2 - s_1|$
The diameter of the capillary tube, and hence its radius, are measured using the travelling microscope. $\gamma = \frac{hr\rho g}{2cos\theta}$

Note that though out the experiment, we assume that the weight of the small quantity of the liquid in the meniscus is negligible, and that temperature remains constant.

Pressure difference in a bubble (Excess pressure within a bubble)



Consider an air bubble which is spherical and of radius r formed inside a liquid of surface tension γ. In the figure shown F is the surface tensional force, P_1 is the external pressure acting on the bubble, P_2 is the internal pressure acting inside the bubble.
Let the cross sectional area, A of the bubble be πr^2.

$$\gamma = \frac{F}{l} \implies F = \gamma l \text{ but } l = 2\pi r \qquad \therefore F = 2\pi r\gamma$$

When the bubble is in equilibrium,

Force due to P_1 + Surface tensional forces = Force due to P_2

$$P_1A + 2\pi r\gamma = P_2A \implies P_1\pi r^2 + 2\pi r\gamma = P_2\pi r^2$$

$$\therefore \pi r^2(P_2 - P_1) = 2\pi r\gamma \implies P_2 - P_1 = \frac{2\gamma}{r}$$

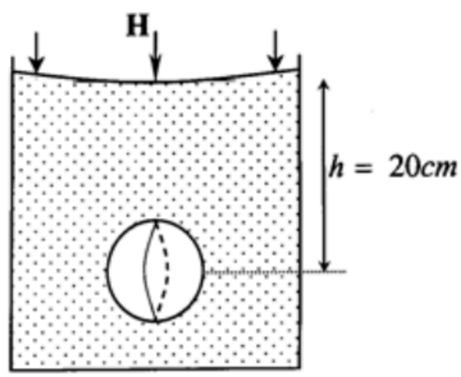
$P_2 - P_1$ is the pressure difference.

Therefore pressure difference across an air bubble is $\frac{2\gamma}{r}$

Example

Calculate the pressure inside a spherical air bubble of diameter 0.1cm blown at a depth of 20cm below the surface of a liquid of density $1.26\times 10^3 kgm^{-3}$ and surface tension $0.064Nm^{-1}$.
(Given: Height of mercury barometer is and density of mercury is $13.6\times 10^3 kgm^{-3}$)

Solution



P_2 = internal pressure, P_1 = external pressure

H = (height of barometer) $\times \rho_{mercury} \times$ g

$= 13.6\times 10^3 \times 0.76 \times 9.81$

$= 101396.2$pa

Pressure due to liquid column = h ρ_{liquid}g

$= (20 \times 10^{-2}) \times 1.26 \times 10^3 \times 9.81$

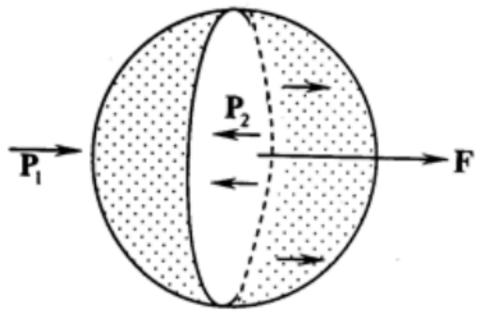
$= 247.212$ pa

$\therefore$ Total external pressure, $P_1 = 247.212 + 101396.2 = 101643.4$pa

Excess pressure, $P_2 - P_1 = \frac{2\gamma}{r} \implies P_2 = P_1 + \frac{2\gamma}{r}$

$$P_2 = 101643.4 + \frac{2 \times (0.064)}{0.05 \times 10^{-2}} = 1.02 \times 10^5 pa$$

Pressure Difference across a soap bubble in air



A soap bubble has two liquid surfaces in contact with air one inside the bubble and the other outside. Therefore, surface tensional forces

$= 2\times (2\pi r\gamma) = 4\pi r\gamma$

If the bubble is in equilibrium,

Force due $P_1 + Surface\ tensional\ forces$ = Force due to P_2

$P_1 A + 4\pi r\gamma = P_2 A \implies P_1 \pi r^2 + 4\pi r\gamma = P_2 \pi r^2$

$\pi r^2 (P_2 - P_1) = 4\pi r\gamma \implies P_2 - P_1 = \frac{4\gamma}{r}$

$P_2 - P_1$ is the pressure difference.

Therefore pressure difference across soap bubble is $\frac{4\gamma}{r}$

Note that the pressure inside a soap bubble is greater than the outside (external pressure due to air), otherwise the combined effect of the external pressure and the surface tension forces in the soap film would cause the bubble to collapse. (The same argument can be extended to an air bubble)

Example

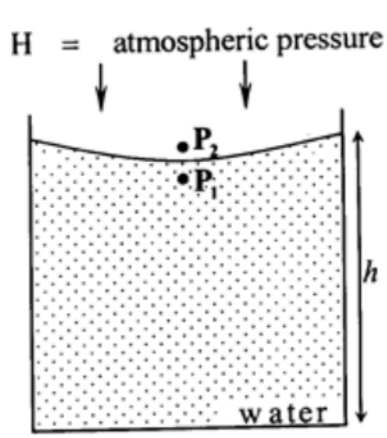
A soap bubble has a diameter of 4mm. calculate the pressure inside it, if the atmospheric pressure is $10^5\ Nm^{-2}$ and that the surface tension of soap solution is $2.8 \times 10^{-2} Nm^{-1}$.

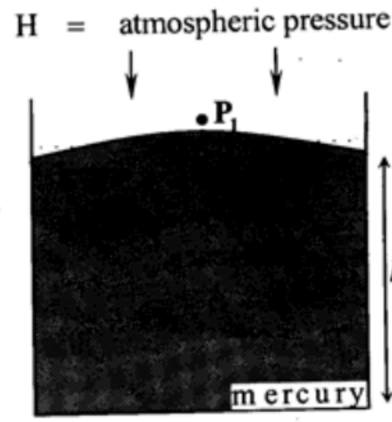
Solution

$P_2 - P_1 = \frac{4\gamma}{r} \implies P_1 + \frac{4\gamma}{r} = P_2$

$$\therefore P_2 = 10^5 + \frac{4 \times (2.8 \times 10^{-2})}{2 \times 10^{-3}} = 100056 pa$$

Pressure difference across a spherical surface





Consider the two situations as illustrated.
The pressure on the concave side of each liquid surface exceeds the pressure on the convex side by $\frac{2\gamma}{r}$ $where\ r$ is the radius of curvature of the surface.

For water: $P_2 - P_1 = \frac{2\gamma cos\theta}{r}$ $but\ P_2 = H\ and\ P_1 = H - h\rho g$

$$\therefore H - (H - h\rho g) = \frac{2\gamma cos\theta}{r} \Rightarrow h\rho g = \frac{2\gamma}{r}$$

$$h = \frac{2\gamma}{\rho g r} \quad (Iff\ \theta = 0)$$

If $\theta \neq 0$, h = $\frac{2\gamma cos\theta}{\rho g r}$ where θ is the angle of contact.

For mercury:

$$P_2 - P_1 = \frac{2\gamma cos\theta}{r} \quad but\ P_1 = H\ and\ P_2 = H + h\rho g$$

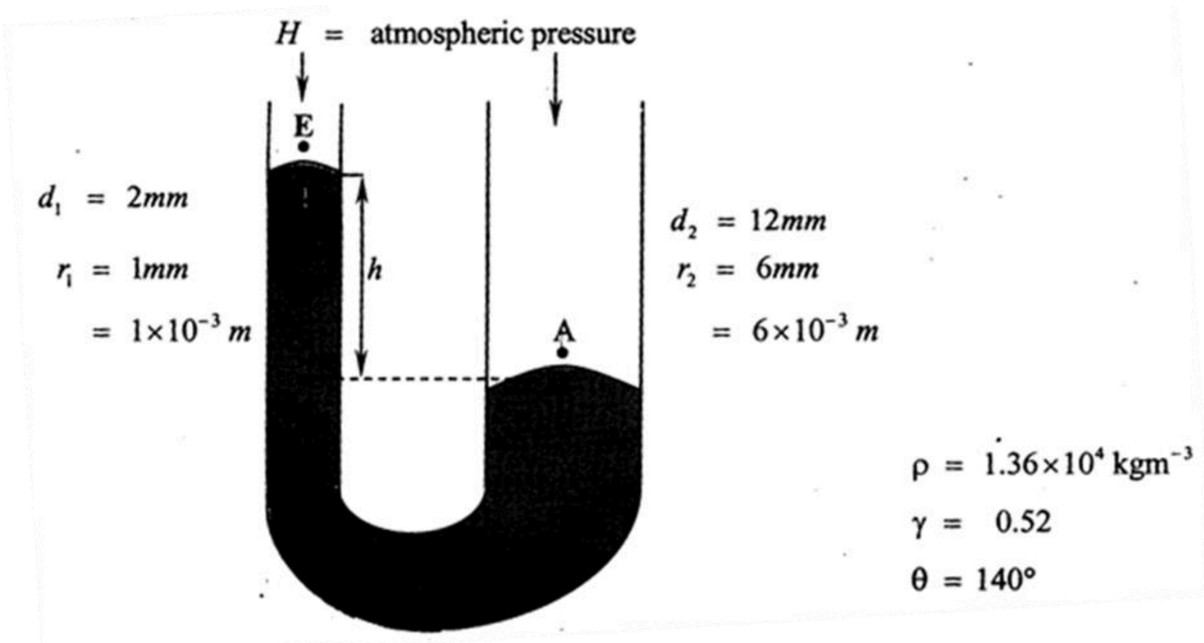
$$\therefore (H + h\rho g) - H = \frac{2\gamma cos\theta}{r} \Rightarrow h = \frac{2\gamma cos\theta}{\rho g\, r}$$

Examples

1. Mercury is poured into a glass U -tube with vertical limbs of diameters 2.0mm and 12.0mm respectively. If the angle of contact between mercury and the glass is 140^0 and the surface tension of mercury is $0.52Nm^{-1}$, calculate the difference in the levels of mercury. (Density of mercury $1.36\times 10^4)\ kgm^{-3})$

Solution

$$P_B - P_A = \frac{2\gamma cos\theta}{r_2} \Rightarrow P_B - P_A + \frac{2\gamma cos\theta}{r_2}$$



Since B and C are at the same level,

$P_C = P_B \;\therefore\; P_c = P_A + \frac{2\gamma cos\theta}{r_2}$ ……………… i)

Also, $P_D - P_E = \frac{2\gamma cos\theta}{r_1} \Longrightarrow P_D = P_E + \frac{2\gamma cos\theta}{r_1}$ ………….. ii)

But, $P_c - P_D = h\rho g \Longrightarrow \left[P_A + \frac{2\gamma cos\theta}{r_2}\right] - \left[P_E + \frac{2\gamma cos\theta}{r_1}\right] = h\rho g$

Also, $P_A = P_E = H \Longrightarrow \left[\frac{2\gamma cos\theta}{r_2}\right] - \left[\frac{2\gamma cos\theta}{r_1}\right] = h\rho g$

$$\therefore h = \left[\frac{2\gamma cos\theta}{\rho g r_2}\right] - \left[\frac{2\gamma cos\theta}{\rho g r_1}\right]$$

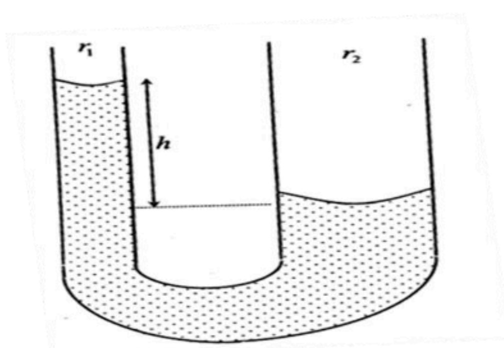
$$= \frac{2\times0.52\times cos140}{(1.36\times10^4)\times9.81\times(6\times10^{-3})} - \frac{2\times0.52\times cos140}{(1.36\times10^4)\times9.81\times(1\times10^{-3})}$$

$$= 4.9 \times 10^{-3} m$$

2. A U-tube with limbs of diameter 7mm and 4mm contains water of surface tension $7 \times 10^{-2}\ Nm^{-1}$ angle of contact 0 and density 1000kgm^{-3}. Find the difference in the levels.

Solution

The reader is advised to follow the steps as those taken in the previous example to show that for this case,



$h\rho g = \left[\frac{2\gamma cos\theta}{r_1}\right] - \left[\frac{2\gamma cos\theta}{r_2}\right]$

$\therefore\ \theta = 0, h\rho g = \left[\frac{2\gamma}{r_1}\right] - \left[\frac{2\gamma}{r_2}\right]$

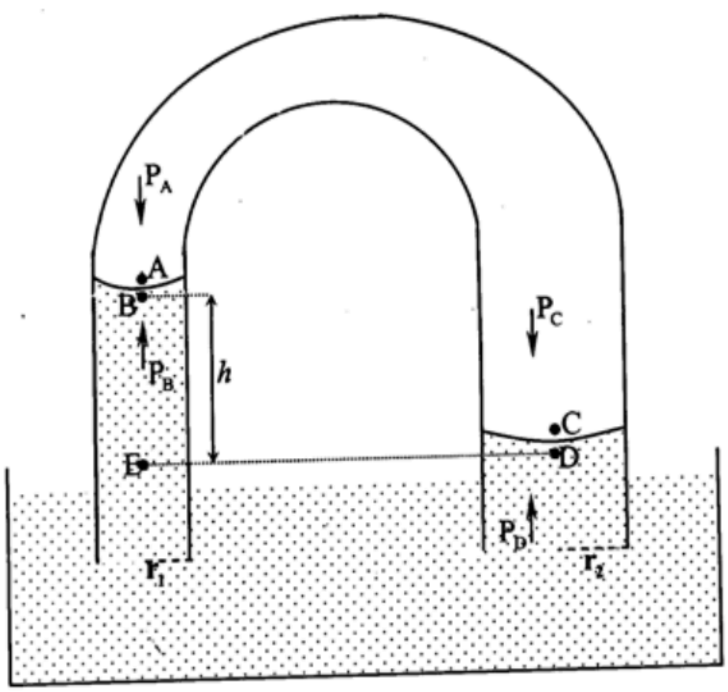
$h = \frac{2\gamma}{\rho g}\left[\left(\frac{1}{r_1}\right) - \left(\frac{1}{r_2}\right)\right]$

$= \frac{2\times(7\times10^{-3})}{1000\times9.81}\left[\left(\frac{1}{2\times10^{-3}}\right) - \left(\frac{1}{3.5\times10^{-3}}\right)\right]$

h = 3.1mm

3. A glass U-tube is such that the diameter of one limb is 4.0mm while that of the other is 8.0mm. The tube is inverted vertically with the open ends below the surface of water in a beaker. Given that surface tension of water is $7 \times 10^{-2}\ Nm^{-1}$, angle of contact between water and glass is zero, and that density of water is 1000kgm^{-3}, what is the difference between the heights to which water rises tin the two limbs

Solution



$r_1 = 2mm = 2\times10^{-3}m$

$r_2 = 4mm = 4\times10^{-3}m$

$\gamma = 7.2\times10^{-2}\ Nm^{-1}$

$\theta = 0°$

For the small limb: $(P_A - P_B) = \frac{2\gamma}{r_1} \Rightarrow P_A = P_B + \frac{2\gamma}{r_1}$ ………. i)

For the large limb: $(P_C - P_D) = \frac{2\gamma}{r_2} \Rightarrow P_C = P_D + \frac{2\gamma}{r_1}$ ………. ii)

But $P_A = P_C \Rightarrow P_B + \frac{2\gamma}{r_1} = P_D + \frac{2\gamma}{r_2}$ …………………………….iii)

$P_D = P_B = 2\gamma\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ $but\ P_D = P_E\ \ and\ P_E - P_B = h\rho g$

$\therefore\ h\rho g = 2\gamma\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \Rightarrow h = \frac{2\gamma}{\rho g}\left[\left(\frac{1}{r_1}\right) - \left(\frac{1}{r_2}\right)\right]$

$= \frac{2\times(7.2\times10^{-2})}{1000\times9.81\times10^{-3}}\left[\left(\frac{1}{2}\right) - \left(\frac{1}{4}\right)\right]$

$= 7.34$mm

4. The internal diameter of the glass tube of a mercury barometer is 3.5mm. The barometer reads 752.4mmHg. Find the correct reading of the barometer after allowing for the error due to surface tension. Density of mercury is 13600kgm^{-3}, its surface tension is 0.52Nm^{-1}, and its angle of contact is 140^0

Solution

The depression due to surface tension, h = $\frac{2\gamma cos\theta}{\rho gr}$

$$= \frac{2\times0.52\times cos140}{(1.75\times10^{-3})\times13600\times9.81} = -3.41\times10^{-3}$$

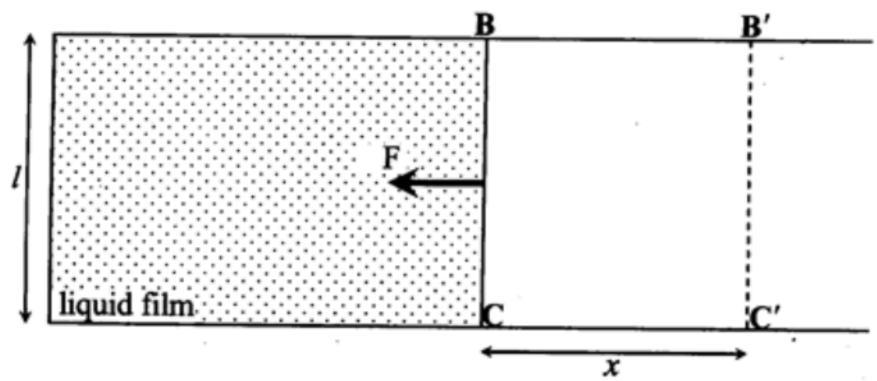
The negative sign indicates that the barometer reading is lower by 3.41mm.

∴ The correct reading of the barometer is 752.4 + 3.41 = 755.81 mmHg.

Surface Energy

This is the amount of work done to produce a fresh surface of liquid film of area 1m^2.

Relationship between surface tension and surface energy



Consider a liquid film stretched on a rectangular metal frame. Supposed that the film is stretched isothermally from BC to *B'C'* through a distance *x* against the surface tensional force, F so that the surface area of the film increases. Surface tensional force = γl. However since there are two liquid surfaces in contact with air, F = 2 × $(\gamma l) = 2\gamma l$

Work done in enlarging the surface force distance = (Force× $(distance) = (2\gamma l)\times(x)$

$= \gamma\ (2\gamma l)$

But, $(2\gamma l)$ is the total increase in the surface area of the liquid film.

∴ Work done = $\gamma\times\Delta A \quad\Rightarrow\quad \gamma = \frac{workdone}{change\ in\ surface\ area}$

We can therefore have an alternative definition of surface tension as:

Surface tension is the work done in enlarging a surface of a liquid film by1m^2 under isothermal conditions.

An alternative SI unit of surface tension is Jm^{-2}.

Note: that this work done is stored as surface energy in the liquid film.

Examples

1. Calculate the amount of work done in breaking up a drop of water of radius 0.5cm into tiny droplets of water, each of radius 1mm, assuming isothermal conditions. Also determine the number of droplets formed given that the surface tension of water is $7 \times 10^{-2}\ Nm^{-1}$.

Solution

$R = 0.5\text{cm} = 5 \times 10^{-2}\ m\ and\ r = 1mm = 1 \times 10^{-3}m.$

Let n be the number of droplets formed

Volume of big drop $= \frac{4}{3}\pi R^3 = \frac{4}{3} \times \frac{22}{7} \times (5 \times 10^{-3})^2 = 5.23 \times 10^{-7}\ m^3$

Total volume of tiny droplets = n $\times \left[\frac{4}{3}\ \pi r^3\right] = n\ (4 \times 10^{-9})$.

But the volume remains the same; $\Rightarrow\ 5.23 \times 10^{-7} = n\ (4 \times 10^{-9})\ \therefore\ n = 125$

Therefore, there are 125 droplets.

Surface area of big droop $4\pi R^2 = 4 \times \frac{22}{7} \times (5 \times 10^{-2}) = 3.14 \times 10^{-4})m^2$.

Surface area of 125 small droplets $= 125 \times (4\pi r^2\ ) = 0.01257m^2$

Change in area, $\Delta A = (0.01257 - 3.14 \times 10^{-4}\ ) = 1.23 \times 10^{-2}m^2$.

But work done = (surface tension) × (change in surface area) = $(7 \times 10^{-2}) \times (\ 1.23 \times 10^{-2})$

$= 8.6 \times 10^{-4}\ Joules.$

2. A liquid drop of diameter 0.5cm breaks up into 27 tiny droplets all of the same size. If the surface tension of the liquid is 0.07 Nm^{-1}, calculate the resulting change in energy.

Solution

Diameter = 0.5cm $\Rightarrow r = 0.25cm = 2.5 \times 10^{-3}m, n = 27$

Let the radius of the tiny droplets be x.

Volume of tiny droplets $= n\left(\frac{4}{3}\pi x^3\right)$ and volume of big drop $= \frac{4}{3}\pi r^3$

$$\therefore\ 27 \times \left(\frac{4}{3}\pi x^3\right) = \frac{4}{3}\pi \times (2.5 \times 10^{-3})^3 \Rightarrow x = 8.3 \times 10^{-4}m$$

Surface area of big drop $= 4\pi r^2 = 4 \times \frac{22}{7} \times (2.5 \times 10^{-3})^2 = 7.85 \times 10^{-5}m^2$.

Surface area of small droplets $= 27 \times (4\pi x^2) = 27 \times [4 \times \frac{22}{7} \times (8.3 \times 10^{-4})^2]$

$= 2.34 \times 10^{-4}m^2.$

Change in surface area, $\Delta A = (2.34 \times 10^{-4}) - (7.85 \times 10^{-5}) = 1.55 \times 10^{-4})m^2$

But work done = (surface tension) × (change in surface area)

$= (7 \times 10^{-2}) \times (\ 1.55 \times 10^{-4}) = 1.09 \times 10^{-5}$ joules.

The change in energy is equal to the work done = 1.09×10^{-5} joules.

3. Calculate the change in surface energy of a soap bubble when its radius decreases from 5cm to 1cm, given that the surface tension of soap solution is $2 \times 10^{-2})Nm^{-1}$.

Solution

Initial area $= 4\pi R^2 = 4 \times \frac{22}{7} \times (5 \times 10^{-2}\ )^2 = 0.0314m^2$

Final area $= 4\pi r^2 = 4 \times \frac{22}{7} \times (1 \times 10^{-2}\ )^2 = 0.001257m^2$

Change in surface area = $0.03014m^2$

Since it's a soap bubble; change in surface area $2 \times (0.03014) = 0.06029m^2$

But work done = (surface tension) × (change in surface area) = $(2 \times 10^{-2}) \times (0.06029)$

$= 1.21\times 10^{-3}$ *joules.*

∴ Change in surface energy = 1.21×10^{-3} *joules.*

Relationship between surface area and shape of a drop

The area of a liquid surface has he least number of molecules in it under surface tensional forces.

Surface area of a given volume of a liquid is therefore a minimum, and according to mathematics, the shape of a given volume of a liquid with minimum surface area is a sphere. This is why the meniscus and small droplets of a mercury and rain drops are spherical in shape approximately.

Small mercury droplets are spherical while large ones flatten out

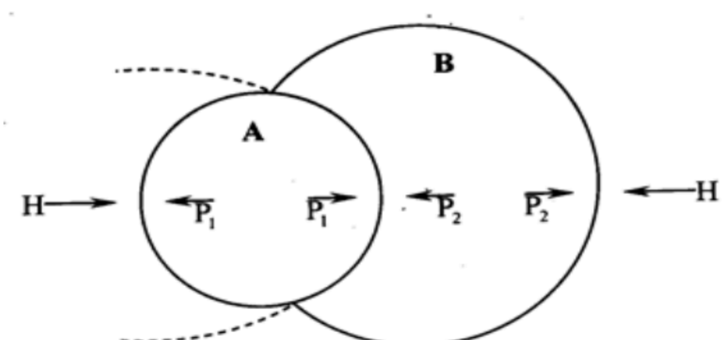
A small drop takes on a spherical shape to minimize the surface energy which tends to be greater than the gravitational potential energy. Therefore the gravitational force cannot distort the spherical shape due to the very small mass of tiny droplets.

A large drop flattens out in order to minimize the gravitational potential energy, which tends to exceed the surface energy. Due to its large weight, gravitational force distorts the spherical shape of large drops. The shape of the drop must agree with the principle that the sum of gravitational potential energy and surface energy must be a minimum.

COMBINED BUBBLES

1. **Case 1**

Consider two soap bubbles A and B of radii r_1 and r_2, where $r_2 > r_1$. If the two soap bubbles come into contact and have a common interface, then the radius of curvature, r of the common interface can be calculated using pressure differences.



For A: $P_1 - H = \frac{4\gamma}{r_1}$ ⟹ $P_1 = H + \frac{4\gamma}{r_1}$

For B: $P_2 - H = \frac{4\gamma}{r_2}$ ⟹ $P_2 = H + \frac{4\gamma}{r_2}$

Since the interface is convex towards B, $P_1 > P_2$ ∴ $P_1 - P_2 = \frac{4\gamma}{r}$

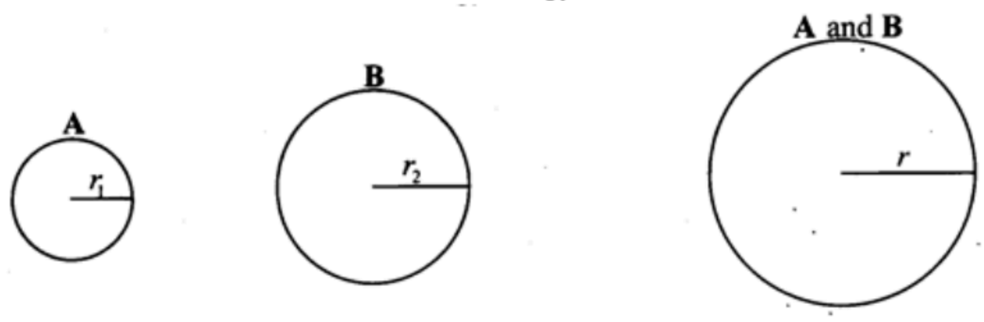
Substituting for P_1 and P_2 gives: $\left(H + \frac{4\gamma}{r_1}\right) - \left(H + \frac{4\gamma}{r_2}\right) = \frac{4\gamma}{r}$

$\therefore \ \frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_2} \ or \ r = \frac{r_1 r_2}{r_2 - r_1}$

$\therefore$ Pressure difference $= \frac{4\gamma}{r} = \frac{4\gamma\,(r_2 - r_1)}{r_1 r_2}$

2. Case 2

Consider two soap bubbles A and B of radii r_1 and r_2 which come together and coalesce to form a single bubble. To find the radius, r of the common interface of the resulting soap bubble, we use conservative of surface energy.



For A: surface energy = $2 \times [\,(4\pi r_1^2) \times \gamma\,]$

For B Surface energy $= 2 \times [\,(4\pi r_2^2) \times \gamma\,]$

When the drops are combined, surface energy = $2 \times [\,(4\pi r^2) \times \gamma\,]$

By conservation of energy, $8\pi r_1^2\gamma + 8\pi r_2^2\gamma = 8\pi r^2\gamma$

$\therefore \ r_1^2 + r_2^2 = r^2 \ \Rightarrow \ r = \sqrt{r_1^2 + r_2^2}$

$\therefore$ Pressure difference $= \frac{4\gamma}{r} = \frac{4\gamma}{r_1^2 + r_2^2}$

Examples

1. Two soap bubbles of radii 2cm and 4cm respectively coalesce under isothermal conditions. If the surface tension of the soap solution is $2 \times 10^{-2}\ Nm^{-1}$, Calculate the excess pressure inside the resulting soap bubble.

Solution

$r = \sqrt{2^2 + 4^2} = \sqrt{20}$ cm $= \sqrt{20} \times 10^{-2}\ m$

Excess pressure = $\frac{4\gamma}{r} = \frac{4\times(2\times10^{-2})}{\sqrt{20}\times10^{-2}} = 1.789\ pa$

2. Two soap A and B of radii 67cm and 10cm respectively coalesce so as to have a portion of their surfaces in common. Calculate the radius of curvature of this common surface and hence the pressure difference.

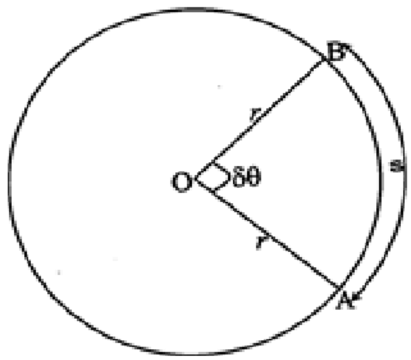
Solution

$r = \frac{r_1 \; r_2}{r_2 - r_1} = \frac{10 \times 6}{10 - 6} = 0.15\text{m}$

Pressure difference $= \frac{4\gamma}{r} = \frac{4 \times (2.5 \times 10^{-2})}{0.15} = 0.667pa.$

CHAPTER 10: CIRCULAR MOTION

Circular motion is the motion of an object moving in a circular path with uniform speed about a fixed point. Consider a particle of mass of mass m moving along the circumference of a circle centre O, and radius r as shown in the diagram.



If the particle covers a distance s in a time t, then its speed v is given by;

$$v = \frac{s}{t}$$

As the particle moves a distance s, it goes through an angle $\delta\theta$. If $\delta\theta$ is in in radians, then; s = $r\delta\theta$

$$\Rightarrow v = \frac{r\,\delta\theta}{\delta t} \quad = \quad r\frac{\delta\theta}{\delta t}$$

$$As\ t \longrightarrow 0, \frac{\delta\theta}{\delta t} \longrightarrow \frac{d\theta}{dt} \Rightarrow v = r\frac{d\theta}{dt}$$

But the rate of change of the angle $\delta\theta$ with time, $\left({d\theta}/{dt}\right)$ is called angular velocity, ω

$$\Rightarrow \omega = \frac{d\theta}{dt} \qquad\qquad \therefore v = \omega r \quad or \quad \omega = \frac{v}{r}$$

Angular velocity is the rate of change of the angle for an object moving in a circular or path about the centre.

The SI unit of angular velocity is radians per second

Period, T : is the time taken to make one complete revolution.

For a complete revolution, $s = 2\pi r, and\ v = r\omega\ and\ from\ time = \frac{distance}{speed},\ \Rightarrow\ T = \frac{2\pi r}{r\omega}$

$$\therefore T = \frac{2\pi}{\omega}$$

The SI unit of period is seconds

Frequency, f: is the number of revolutions or cycles made in one second.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Examples:

1. A particle moving in a circular path of radius 50cm has a speed of 12ms^{-1}. Find the angular speed of the particle in
   (i) Radians per second
   (ii) Revolutions per minute

Solution

(i) v = 12ms^{-1} and r = 50cm = 0.5m

But $\omega = \frac{v}{r} = \frac{12}{0.5} = 24\ radians\ per\ second$

(ii) $2\pi\ radians \rightleftharpoons\ 360^0 \Longrightarrow 2\pi\ radians \rightleftharpoons\ 1 revolution$

$\therefore 1 radian \rightleftharpoons \frac{1}{2\pi} revolutions\ \Rightarrow 1\ radian\ per\ second = \frac{1}{2\pi} revolutions\ per\ second$

24 radians per second $= 24 \times \frac{1}{2\pi} revolutions\ per\ second$

$= 24 \times \frac{1}{2\pi} \times 60\ revolutions\ per\ minute$

= 229.18 revolutions per minute.

2. Calculate the velocity of the earth which has a radius of 6400km and a period of 24hours.

Solution

Radius = 6400000m, T = 86400s

From $T = \frac{2\pi}{\omega}$ and $\omega = \frac{v}{r}$,

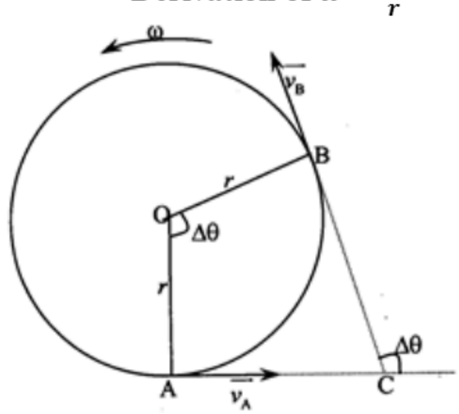
$v = \frac{2\pi r}{T} = \frac{2 \times 6400000 \times \pi}{86400}$

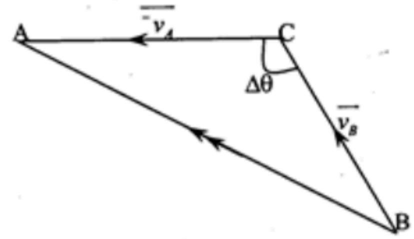
$= 465.42ms^{-1}$

Trial questions

1. A particle moves along a circular path of radius 3m with an angular velocity of 20 $rads^{-1}$. Calculate the:
   (i) Linear speed of the particle
   (ii) Angular velocity in revolutions per second
   (iii) Time taken for one revolution
   [Ans: $v = 60ms^{-1}$, $\omega = 3.2$ rev s^{-1}, T = 0.31s]
2. Winnie and Annah stand on a playground roundabout respectively 50 cm and 175 cm from the centre. The roundabout moves with constant angular velocity and Annah's speed is found to be $1ms^{-1}$. Find
   (i) The angular velocity of the roundabout in rad s^{-1}
   (ii) The time taken for the roundabout to complete ten revolutions
   (iii) Annah's speed
   [Ans: 2 rad s^{-1}, $10\pi s$, 3.5 ms^{-1}]

Derivation of $a = \frac{v^2}{r}$





Suppose a particle of mass m is moving in a circle or a circular path of radius r with a uniform speed v. The velocity at A is $\overrightarrow{v_A}$ in the direction of tangent AC. After a short time, $\Delta t, the$ particle is the particle is at B, and is moving with a velocity $\overrightarrow{v_B}$ in the direction of tangent CB. Since the velocities are in different directions, they are different.

$$BC = |\overrightarrow{v_B}\,| = v, CA = |\overrightarrow{-v_A}| = v \;\; and\; \overrightarrow{BA} \text{ is the change in velocity}$$

$$\therefore acceleration, a = \frac{change\ in\ velocity}{time\ taken}$$

$$\Rightarrow a = \frac{\overrightarrow{BA}}{\Delta t}$$

If Δt is small, then $\angle BCA = \Delta\theta$ is also small, and since Ba is approximately an arc,

$$\therefore BA = v\Delta\theta \;\Rightarrow a = v\frac{\Delta\theta}{\Delta t}$$

$$\text{As } \Delta t \longrightarrow 0, \frac{\Delta\theta}{\Delta t} \longrightarrow \frac{d\theta}{dt} \text{ but } \frac{d\theta}{dt} = \omega$$

$$\therefore a = v\omega \;\; but\; \omega = \frac{v}{r}$$

$$\Rightarrow a = \frac{v^2}{r} = r\omega^2$$

Note: This acceleration is directed towards the centre of the circular path.

Centripetal force

This is the force which keeps a body moving in a circular path and is directed towards the centre of the circular path.

From F = ma, if F is centripetal force, then $a = \frac{v^2}{r} = r\omega^2$

$$F = \frac{mv^2}{r} = mr\omega^2$$

Example

A body of mass 2kg moves with a constant angular speed of 5 rad s^{-1} around a circle of radius 10cm. Find the magnitude of the force that must be acting on the body towards the centre of the circle.

Solution

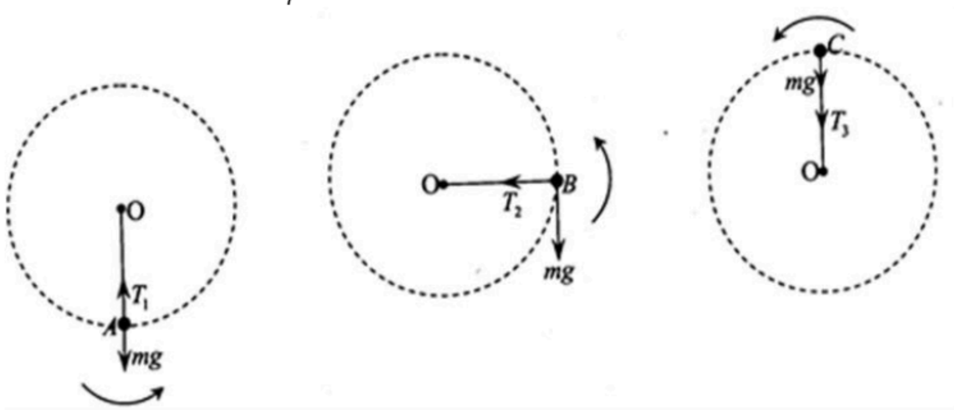
$$F = mr\omega^2 = 2 \times 0.1 \times 5^2 = 5N$$

Trial questions

1. A particle of mass 0.2kg moves in a circular path with an angular velocity of 5 rad s^{-1} under the action of a centripetal force of 4N. What is the radius of the paths? [Ans: 0.8m]
2. What force is required to cause a body of mass 3g to move in a circle of radius 2m at a constant rate of 4 revolutions per second? [Ans: 3.8N]

Vertical Circle

Consider a body of mass m whirled in a vertical circle O by a string of length *l,* equal to the radius r of the circle moving with a constant speed v. Also consider the three positions of the body as it moves in a circle. The tension in the string varies as the body moves. It should however be noted that in whatever case, the resultant force towards the centre of the circular path is the centripetal force, $F = \frac{mv^2}{r}$



At A:

Resultant force towards the centre = mg – T_1

$$\Rightarrow \frac{mv^2}{r} = mg - T_1 \quad \therefore T_1 = \frac{mv^2}{r} + mg$$

At B:

Resultant force towards the centre = T_2

$$\Rightarrow \frac{mv^2}{r} = T_2 \qquad \therefore T_2 = \frac{mv^2}{r}$$

At C:

Resultant force towards the centre = mg + T3

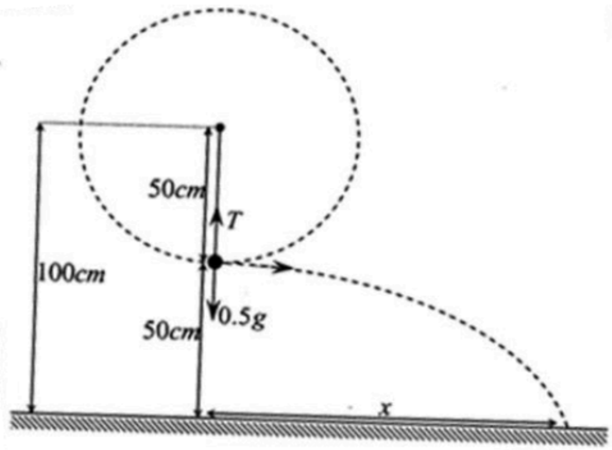
$$\Rightarrow \frac{mv^2}{r} = mg + T_3 \qquad \therefore T_3 = \frac{mv^2}{r} - mg$$

It can therefore be noted that tension in the string is greatest when the body is at the bottom of the circle. It's at the same position that the string is most likely to break

Examples

1. A stone of mass 500g is attached to a string of length 50cm which will break if the tension in it exceeds 20N. The string is whirled in a vertical circle, the axis of rotation being at a height of 100cm above the ground. The angular speed is gradually increased until the string breaks.
   (i) In what position is this break likely to take place, and at what angular velocity?
   (ii) Where will the stone hit the ground?

Solution



(i) The string is most likely to break at the bottom of the circle since it's at this point that the string has maximum tension in it.

$$T = \frac{mv^2}{r} + mg = mr\omega^2 + mg$$

m = 0.5kg, r = 0.5m

$$\therefore 20 = 0.5 \times 0.5 \times \omega^2 + 0.5 \times 9.81$$

$$\omega^2 = 60.38$$

$$\omega = 7.8\, rad\, s^{-1}$$

(ii) Consider the motion of the body when the spring breaks,

$$y = ut\sin\theta + \frac{1}{2}gt^2$$

y = 0.5m, $\theta = 0$, $\Rightarrow 0.5 = \frac{1}{2} \times 9.81 \times t^2 \Rightarrow t = 0.32s$

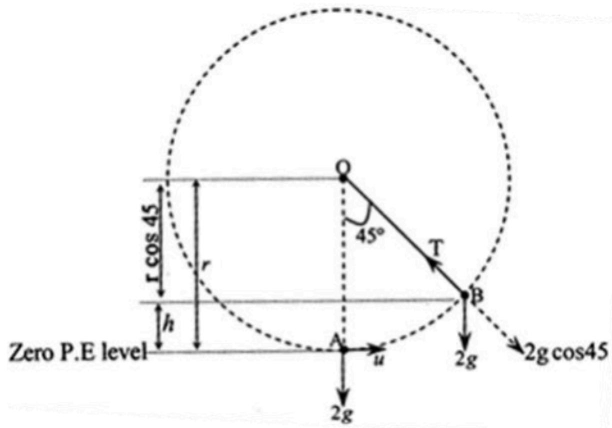
Also $x = ut\cos\theta \Rightarrow x = u \times 0.32 \times \cos 0 = 0.32u$

But $\omega = \frac{v}{r} \Rightarrow v = r\omega = 0.5 \times 7.8$

$$\therefore x = 0.5 \times 7.8 \times 0.32 = 1.25m$$

2. A particle of mass 2kg is suspended from a fixed point O by a light inextensible string of length 20cm. The particle is projected from the lowest point A with a horizontal speed of 5ms^{-1} and describes a vertical circle. When the particle is at point B, then the tension in the string is T N, where OB makes an angle of 45^0 with the downward vertical. Determine the speed of the particle and the tension in the string at B.

Solution



At A:

Kinetic energy $= \frac{1}{2}mu^2 = \frac{1}{2} \times 2 \times 5^2$ $\quad$ potential energy = mgh

= 25J $\quad$ since h = 0, P.e =0

$\Rightarrow kinetic\ enegy + potential\ energy = 25J$

At B:

Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times v^2$ $\quad$ potential energy = mgh but h = r – r cos 45

$= v^2$ $\quad$ ⇒ P.e = 2g(r – r cos 45)

$= 2g\left(0.2 - 0.2 \times \frac{1}{\sqrt{2}}\right)$ = 1.15J

$\Rightarrow kinetic\ enegy + potential\ energy = 1.15 + v^2$

From the principle of conservation of mechanical energy, total energy at A should be equal to total energy at B, thus

$25 = 1.15 + v^2$

$\Rightarrow v = \sqrt{25 - 1.15} = 4.88\text{ms}^{-1}$

Also, from the figure; Net force towards the centre = $T - mg\cos\theta$

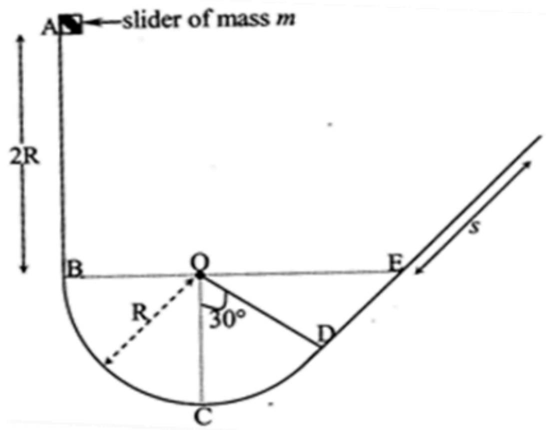
$\frac{mv^2}{r} = T - 2g\cos 45$

$\therefore T = \frac{2\times 4.88^2}{r} - 2 \times 9.81 \times \frac{1}{\sqrt{2}} = 252.02N$

Trial questions

1. A bucket of water is swung in a vertical circle of radius 64m in such a way that the bucket is upside down when it is at the top of the circle. What is the minimum speed that the bucket may have at this point if the water is to remain in it? [Ans: 25.06 ms^{-1}]
2. An aero plane loops the loop in a vertical circle of radius 200m with a speed of 40ms^{-1} at the top of the loop. The pilot has a mass of 80 kg. What is the tension in the strap holding him into his seat when he is at the top of the loop? [Ans: 60N]
3. In the figure below, ABCDE, is a truck in a vertical plane. The track is straight from A to B, circular from B to D, and then straight from D onwards. O is the centre of the circular paths, and R is the radius. The track is smooth from A to E and rough from E onwards

(coefficient of kinetic friction, μ). A small slider of mass m is released from rest while in position A and then slides along the track.



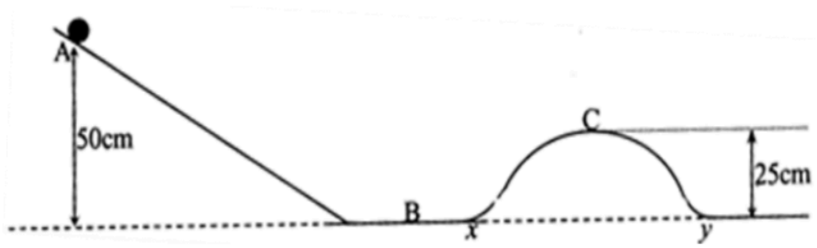
Determine the:

(i) Normal force NB exerted by the track on the slider just after it passes point B,

(ii) Normal force NC exerted by the track on the slider as it passes the bottom point C

(iii) Show that the distance s traveled along the incline past point E before the slider stops is given by $s = \frac{4R}{1+\mu\sqrt{3}}$ [Ans: NB = 4mg, NC = 7mg]

4.



In the figure above, ABC is a smooth track in a vertical plane. The curved section xCy is a circular arc of radius of curvature 75cm. A ball bearing of mass 60g is released from A and it moves along the track throughout its motion. Calculate the;

(i) Speed of the ball bearing at B,

(ii) Speed of the ball bearing at C

(iii) Reaction between the track and the ball bearing at a point C

[$V_B = 3.2\ ms^{-1}$, $V_C = 2.2\ ms^{-1}$, F = 0.2N]

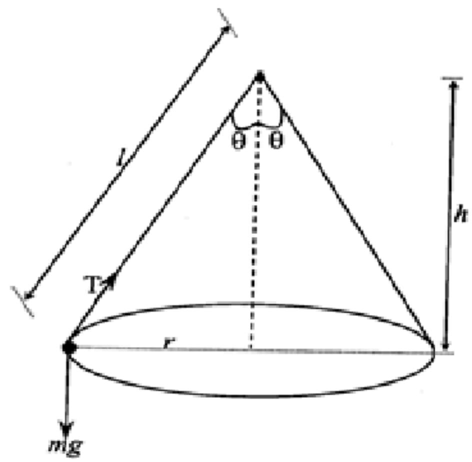
5. A particle of mass 5 kg is slightly disturbed from rest on the top of a smooth hemisphere of radius 4m, and centre O, which has its plane of a horizontal ground.

(i) Show that the particle leaves the surface of the hemisphere at the point P,where the angle between the radius PO and the upward vertical is $\cos^{-1}\frac{2}{3}$

(ii) After leaving the surface of the surface of the hemisphere, the particle hits a vertical wall situated perpendicular to the horizontal diameter through O and at distance of $\frac{5\sqrt{5}}{3}$ m from the centre of the hemisphere. Find the height of the particle above the ground when it hits the wall. [Ans: $\frac{307}{192}m$]

Conical pendulum

Consider a particle of mass m attached to the lower end of a light inextensible string, the upper end of which is fixed as shown in the diagram.



When the particle is whirled in a horizontal circle of radius r, at a constant speed v, the string describes the curved surface of a cone and turns at a constant angle,θ to the vertical. This arrangement is known as a conical pendulum.

There are only two forces acting on the particle, the tension in the string, and the weight of the particle.

Since the tension T is at an angle θ to the vertical, it can be resolved into two components, a horizontal component (Tsin θ) towards the centre of the circular paths which gives the centripetal acceleration, and a vertical component (T cos θ) which should be equal to the weight for equilibrium.

Therefore resolving;

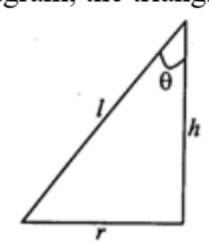
$\rightarrow T\sin\theta$ …….(i) $\qquad \uparrow T\cos\theta = mg$ …………(ii)

Dividing eqn (i) by eqn (ii) gives:

$$\tan\theta = \frac{v^2}{rg}$$

$$\Rightarrow \theta = \tan^{-1}\frac{v^2}{rg}$$

Also from the diagram, the triangle below can be drawn,



From the triangle, $r = l\sin\theta \quad and \;\; h = l\cos\theta$

$$\Rightarrow \sin\theta = \frac{r}{l} \qquad and \qquad \cos\theta = \frac{h}{l}$$

Substituting for $\sin\theta$ in eqn (i) gives:

$T\frac{r}{l} = \frac{mv^2}{r}$ but v = rω

$\Rightarrow T = \frac{ml(r\omega)^2}{r^2} = ml\omega^2$ …………..(iii)

Substituting for $\cos\theta$ in eqn (ii) gives:

$T\frac{h}{l} = mg$

$\Rightarrow T = \frac{mgl}{h}$ ……………(iv)

Equating equations (iii) and (iv) gives:

$ml\omega^2 = \frac{mgl}{h}$

$\Rightarrow \omega = \sqrt{\left(\frac{g}{h}\right)}$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{g}{h}\right)}} = 2\pi\sqrt{\left(\frac{h}{g}\right)}$$

$$\text{Frequency} = \frac{\omega}{2\pi} = \frac{\sqrt{\left(\frac{g}{h}\right)}}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{g}{h}\right)}$$

Examples:

1. For an arrangement as the one above (conical pendulum), show that:

   (i) $v = \left(\frac{r^2 g}{h}\right)^{1/2}$ (ii) $T = \frac{mg}{h}(r^2 + h^2)^{1/2}$

   Solution

   (i) v = rω and $\omega = \sqrt{\left(\frac{g}{h}\right)} \Rightarrow v = r\sqrt{\left(\frac{g}{h}\right)}$

   $\therefore v^2 = r^2\frac{g}{h}$

   $\Rightarrow v = \left(\frac{r^2 g}{h}\right)^{1/2}$

   (ii) $T = ml\omega^2$ but $\omega = \sqrt{\frac{g}{h}}$ $and\ l^2 = r^2 + h^2$

   $\Rightarrow T = m(r^2 + h^2)^{1/2}\frac{g}{h}$

   $= \frac{mg}{h}(r^2 + h^2)^{1/2}$

2. A mass of 0.2kg is whirled in a horizontal circle of radius 0.5m by a string inclined at 30^0 to the vertical. Calculate the :
   (i) Speed of the mass in the horizontal circle
   (ii) Length of the string
   (iii) Vertical height from the point of suspension to the centre of the circle
   (iv) Angular speed
   (v) Period

   Solution

(i) $T\cos\theta = mg$

$\Rightarrow T = \frac{0.2 \times 9.81}{\cos 30}$

(ii) $T \sin\theta = \frac{mv^2}{r}$

$\Rightarrow v^2 = \frac{rT\sin\theta}{m}$

$v = \sqrt{\frac{2.27 \times 0.5 \times \sin 30}{0.2}} = 1.86ms^{-1}$

(iii) $r = l\sin\theta$

$\Rightarrow l = \frac{r}{\sin\theta} = \frac{0.5}{\sin 30} = 1m$

(iv) $h = l\cos\theta$

$= 1\times \cos 30 = 0.866m$

(v) $\omega = \frac{v}{r} = \frac{1.68}{0.5}$

$= 3.36 \text{ rad s}^{-1}$

(vi) Period $= \frac{2\pi}{\omega} = \frac{2\times 3.14}{3.36} = 1.87\text{s}$

3. A pilot banks the wings of an air craft so as to travel at a speed of 540kmh^{-1} along a horizontal circular paths of radius 8km. Calculate:
   (i) The centripetal force
   (ii) Angle the pilot should bank the air craft

Solution

$v = 540kmhr^{-1} = 540\times \frac{1000}{3600} = 150\text{ms}^{-1}$ and r = 8000m

(i) $a = \frac{v^2}{r} = \frac{150^2}{8000} = 2.81\text{ms}^{-2}$

(ii) $\tan\theta = \frac{v^2}{rg} = \frac{150^2}{8000\times 9.81} = 0.2867 \Rightarrow \theta = 16^0$

Trial questions

1. A particle of mass 0.20kg is attached to one end of a light inextensible string of length 50cm. The particle moves in a horizontal circle with an angular velocity of 5.0 rad s^{-1} with the string inclined at an angle θ to the vertical. Find the value of θ

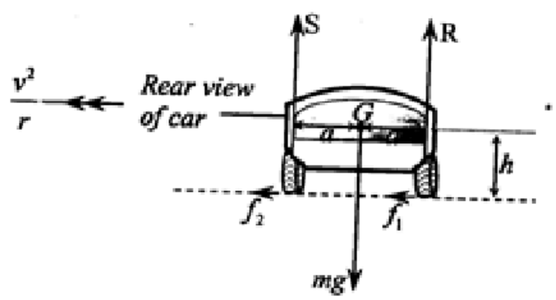
   [Ans: 37^0]
2. A particle of mass 0.25kg is attached to one end of a light inextensible string of length 3.0m. The particle moves in a horizontal circle and the string sweeps out the surface of a cone. The maximum tension that the string can sustain is 12N. Find the maximum angular velocity of the particle. [Ans: 4 rad s^{-1}]]

3. An astronaut, as part of her training, is spun in a horizontal circle of radius 5m. If she can withstand a maximum acceleration of $78.5ms^{-2}$, what is the maximum angular velocity at which the astronaut can remain conscious? [Ans: 3.96 rad s^{-1}]
4. A light inextensible string AB of lengh $2l$ has a particle attached to its midpoint C The ends A and B of the string are fastened to two fixed points with A a distance l vertically above B. With both parts of the string taut, the particle describes a horizontal circle about the line AB with a constant angular speed ω. If the tension in CA is three times that in CB, show that , $\omega = 2\sqrt{\frac{g}{l}}$
5. A pendulum bob of mass 0.2kg is attached to one end of an inelastic string of length 1.2m. The bob moves in a horizontal circle with the string inclined at 30^0 to the vertical. Calculate
   (i) The tension in the string
   (ii) The period of the motion [Ans: (i) 2.27N (ii) 2.04s]

Applications of circular motion

1. **Motion of a car around a circular track**

Consider a car with centre of gravity G which is at a height h above the ground negotiating a bend on a level track. Also assume that the distance between its rear tyres is 2a



For circular motion, the frictional forces f_1 and f_2 at the wheels of the car supply the centripetal force.

$$\therefore f_1 + f_2 = \frac{mv^2}{r} \text{ ....(i)}$$

Since the car does not move off the road, then the sum of upward forces must be equal to the sum of downward forces.

$$\therefore R + S = mg \text{ ....(ii)}$$

Taking moments about the centre of gravity G:

$$f_1 \times h + f_2 \times h + S \times a = R \times a$$

$$\Rightarrow (f_1 + f_2)h = (R - S)a \text{ ....(iii)}$$

Consider equations (i) and (ii),

Substituting for $f_1 + f_2$ in equatioin (iii) gives: $\frac{mv^2}{r}h = (R - S)a$

$$\Rightarrow R - S = \frac{mhv^2}{ar} \text{ ....(iv)}$$

Consider equations (ii) and (iv)

Adding the two equations gives:

$$R + S = mg$$
$$+$$
$$R - S = \frac{mhv^2}{ar}$$
$$2R = mg + \frac{mhv^2}{ar}$$

Subtracting the two equations gives:

$$R + S = mg$$
$$-$$
$$R - S = \frac{mhv^2}{ar}$$
$$2S = mg - \frac{mhv^2}{ar}$$
$$\Rightarrow S = \frac{1}{2}m(g - \frac{hv^2}{ar})$$

It should be noted that if S = 0, then the car is just about to overturn, topple or upset.

$$\Rightarrow 0 = \frac{1}{2}m(g - \frac{hv^2}{ar})$$
$$g - \frac{hv^2}{ar} = 0$$
$$\therefore v^2 = \frac{gar}{h} \Rightarrow v = \sqrt{\frac{gar}{h}}$$

If the car is driven at a velocity $v = \sqrt{\frac{gar}{h}}$, then its at a point of toppling/overturning/upsetting.

Therefore, for no toppling/overturning/upsetting, then $v < \sqrt{\frac{gar}{h}}$

If $v > \sqrt{\frac{gar}{h}}$, then the car overturns/topples/upsets

It can therefore be noted that for upsetting/toppling/overturning;

- The centre of gravity should be high i.e h should be large
- The bend should be sharp i.e r should be small
- The distance between the tyres should be small i.e a should be small

Also if the coefficient of friction between the tyres and the ground is μ,

Then from equation (i), $\mu R + \mu S = \frac{mv^2}{r} \Rightarrow \quad \mu(R + S) = \frac{mv^2}{r}$

Substituting for R + S, $\mu mg = \frac{mv^2}{r} \Rightarrow \ v = \sqrt{\mu rg}$

If the car is driven at a velocity $v = \sqrt{\mu rg}$, then it's at a point of sliding/skidding/slipping.

Therefore, if $v > \sqrt{\mu rg}$, the car slides/skids/slips

It should also be noted that for sliding/skidding/slipping,

- The bend should be sharp i.e r should be small
- The road should be slippery i.e μ should be small
- The vehicle should be moving at a speed $v > \sqrt{\mu rg}$ i.e moving too fast.

Example

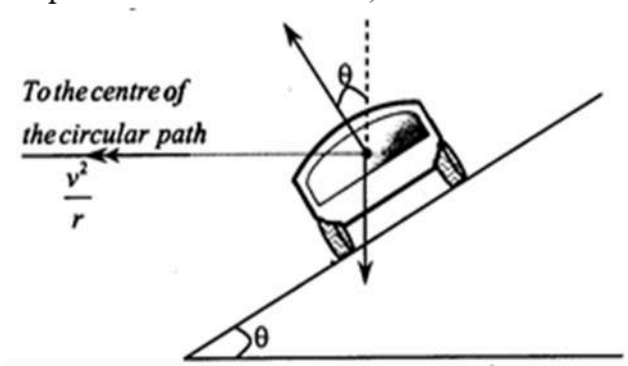
A bend on a level road forms a circular arc of radius 75m. The greatest speed at which the car can travel around the bend without slipping occurring is $63kmh^{-1}$. Find the coefficient of friction between the tyres of the car and the road surface

Solution

$v = 63kmh^{-1} = 63 \times \frac{1000}{3600} = 17.5ms^{-1}$

$v = \sqrt{\mu r g} \Rightarrow \mu = \frac{v^2}{rg} = \frac{17.5^2}{75 \times 9.81} = 0.42$

2. **Motion of a car round a banked or inclined track, with no tendency to skid**.
   Consider a car negotiating a bend inclined at an angle θ to the horizontal. Assuming there is no tendency to slide/skid/slip at the wheels of the car, i.e there is no tendency to slide/skid/slip at the wheels of the car, i.e there are no frictional forces



Resolving: $\rightarrow R \sin\theta = \frac{mv^2}{r}$ ....(i) $\uparrow$ $R \cos\theta = mg$ ....(ii)

Dividing equation (i) by (ii) gives; $\tan\theta = \frac{v^2}{rg}$

$$\Rightarrow v = \sqrt{rg\tan\theta}$$

Example

A car moves around a circular path of radius 75m which is banked at $\tan^{-1}\frac{5}{12}$ to the horizontal. At what speed should the car be driven if it's to have no tendency to slip?

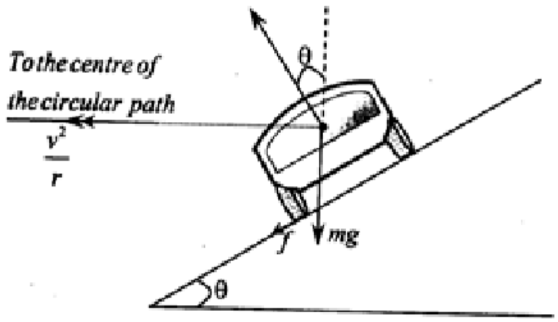
Solution

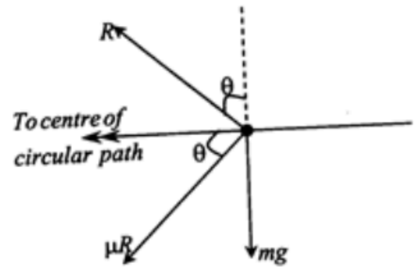
$\Rightarrow v = \sqrt{rg\tan\theta} = \sqrt{75 \times 9.81 \times \frac{5}{12}}$ $=17.51ms^{-1}$.

3. **Motion of a car round a banked track, with a tendency to skid**

Case 1: when speed is maximum

When a car is moving as fast as possible, the maximum frictional force μR acts in such a way to prevent the car from slipping up the plane. This implies that the frictional force acts downwards.





Resolving:

$$\leftarrow R\sin\theta + \mu R\cos\theta = \frac{mv^2}{r} \text{.............(i)}$$

$$\uparrow R\cos\theta = mg + \mu R\sin\theta$$

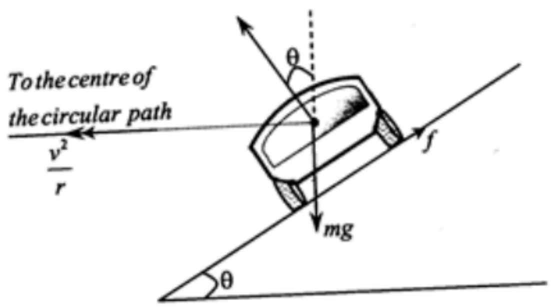
$$\Rightarrow R\cos\theta - \mu R\sin\theta = mg \text{...........(ii)}$$

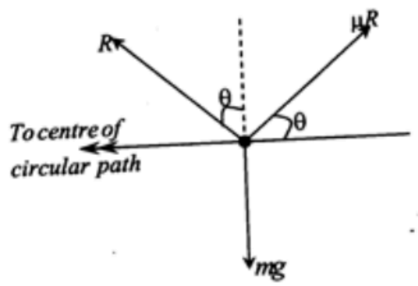
Dividing eqn (i) by eqn (ii) gives:

$$\frac{R\sin\theta+\mu R\cos\theta}{R\cos\theta-\mu R\sin\theta} = \frac{v^2}{rg}$$

Dividing every term by $R\cos\theta$ gives: $\frac{\tan\theta+\mu}{1-\mu\tan\theta} = \frac{v^2}{rg}$

Case 2: when speed is minimum





When the car is moving as slowly as possible, the maximum frictional force μR acts in such a way to prevent the car from slipping down the plane. This implies that the frictional force acts upwards.

Resolving:

$$\leftarrow R\sin\theta - \mu R\cos\theta = \frac{mv^2}{r} \text{.............(i)}$$

$$\uparrow R\cos\theta + \mu R\sin\theta = mg \text{ ....(ii)}$$

Dividing eqn (i) by eqn (ii) gives:

$$\frac{R\sin\theta-\mu R\cos\theta}{R\cos\theta+\mu R\sin\theta} = \frac{v^2}{rg}$$

Dividing every term by $R\cos\theta$ gives: $\frac{\tan\theta-\mu}{1+\mu\tan\theta} = \frac{v^2}{rg}$

Example

A car travels around a bend on a rough road which is a circular arc of radius 62.5m. The road is banked at $\tan^{-1}\frac{5}{12}$ to the horizontal. If the coefficient of friction between the tyres of the car and the road surface is 0.4. Find the:

(i) Greatest speed at which the car can be driven around the bend without slipping occurs
(ii) Least speed at which the car can be driven around the bend without slipping occurring

Solution

(i) $\frac{\tan\theta+\mu}{1-\mu\tan\theta} = \frac{v^2}{rg} \Rightarrow v^2 = \frac{(\tan\theta+\mu)rg}{1-\mu\tan\theta}$

$$= \frac{(\frac{5}{12}+0.4)\times 62.5\times 9.81}{1-0.4\times\frac{5}{12}}$$

$= 600.9$

$\therefore v_{max} = 24.5\text{ms}^{-1}$

(ii) $\frac{\tan\theta-\mu}{1+\mu\tan\theta} = \frac{v^2}{rg} \Rightarrow v^2 = \frac{(\tan\theta-\mu)rg}{1+\mu\tan\theta}$

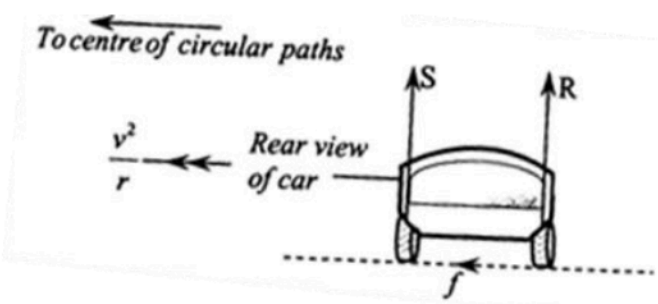
$$= \frac{(\frac{5}{12}-0.4)\times 62.5\times 9.81}{1+0.4\times\frac{5}{12}}$$

$= 7.2$

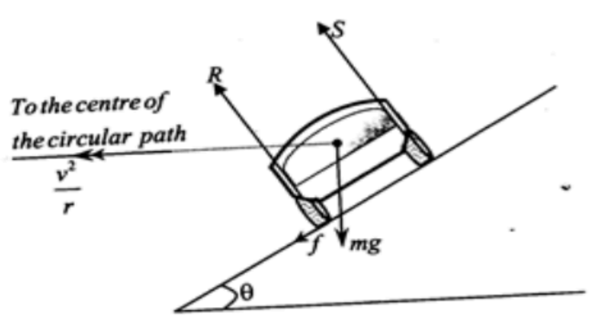
$\therefore v_{min} = 2.7\text{ms}^{-1}$

Notes:

1. At this point we should be in position to explain why a racing car can travel faster on a banked track than on a flat track of the same radius of curvature.



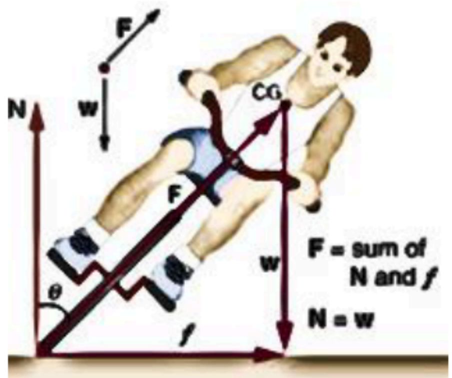
When the track is flat/unbanked, its only the frictional force f acting on the wheels that provides the centripetal force such that when the liming frictional force is exceeded, slipping occurs.



On the other, when the track is banked, the normal reaction has a component, $R\sin\theta$ towards the centre of the circular paths. Therefore the centripetal force is provided by the friction together with this component of the normal reaction (centripetal force = f + $R\sin\theta$).

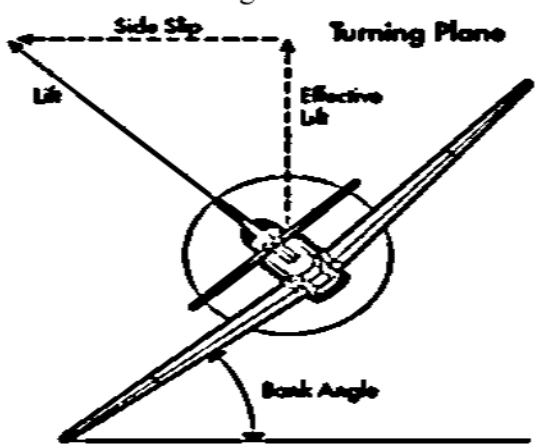
This centripetal force is larger than when the track is flat, and so a racing car can travel faster on a banked track than on a flat truck.

2. We should also be able to explain why it's necessary for a bicycle rider moving round a circular path, the frictional force at the ground provides the inward force towards the centre of the circular paths (centripetal force).



This force provides a moment about his centre of gravity. The rider would therefore have a tendency to fall off in a direction away from the centre of the path if this moment is not counterbalanced. However, when the rider leans towards the centre of the path, his normal reaction bears a moment about his centre of gravity, which counter balances the moment due to friction.

3. We should also calculate the angle at which an aircraft should bank to change its course



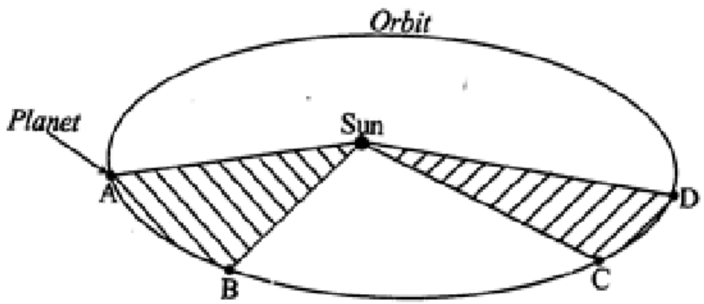
Trial question

1. A car of mass 1000kg moves around a banked track at a constant speed of $108kmh^{-1}$. Assuming the total reaction at the wheels is normal to the track, and the radius of curvature of the track is 100m, calculate the
   (i) Angle of inclination of the track to the horizontal
   (ii) Reaction at the wheels [Ans: (i) 42.53^0, (ii) 13314.08N]

CHAPTER 11: GRAVITATION

Kepler's laws of planetary motion

i. All planets describe ellipses about the sun as one focus
ii. The imaginary line joining the sun and a moving planet sweeps out equal areas in equal time intervals.



Therefore, if the planet takes the same time to move from A to B as from C to D, then the shaded areas should be equal.

iii. The squares of periodic times of revolution of the planets about the sun are directly proportional to the cubes of the mean distance, r from it.

$$\Rightarrow T^2 \propto r^3 \text{ or } T^2 = kr^3 \text{ or } \frac{T^2}{r^3} = constant$$

Newton's law of gravitation

It states that the force between two given bodies is directly proportional to the product of their masses and inversely proportional to the square of their distances apart.

Therefore, if M and m are masses of two particles which are a distance, r , apart, then;

$$F \propto Mm \ldots\ldots(i) \quad and \quad F \propto \frac{1}{r^2} \quad \ldots\ldots(ii)$$

Combining the two equations;

$$F \propto \frac{Mm}{r^2}$$

$$\Rightarrow F = \frac{GMm}{r^2}$$

G is a constant of proportionality known as the universal gravitational constant.

$$G = \frac{Fr^2}{Mm}$$

$\Rightarrow$ *SI* unit of G is Nm^2kg^{-2}

$G = 6.67 \times 10^{-11}\ Nm^2kg^{-2}$

Dimensions of G:

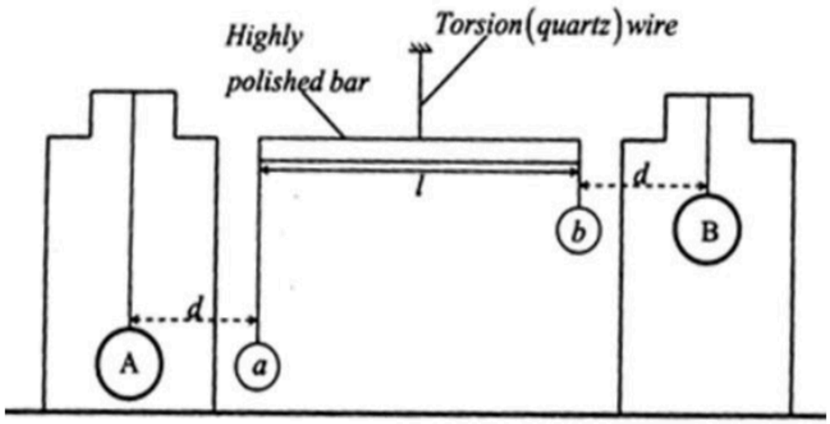
$$[G] = \frac{[F]\times[r^2]}{[M]\times[m]} = \frac{MLT^{-2}\times L^2}{MM} = L^3T^{-2}M^{-1}$$

From the dimensions of G, i.e $[G] = L^3T^{-2}M^{-1}$, it can be seen that G can also be expressed in $m^3s^{-2}kg^{-1}$.

Exercise
Show that the units for G i.e Nm^2kg^{-2} and $m^3s^{-2}kg^{-1}$

However, as seen above, the value of the universal gravitational constant G = 6.67×10^{-11} is small. Therefore, from $= \frac{GMm}{r^2}$, for bodies of small / ordinary masses, the force of attraction is extremely small, and so cannot cause any noticeable motion. This explains why the gravitational force of attraction between two bodies of ordinary masses is not noticeable in everyday life. The only attraction that we notice is that due to acceleration due to gravity of the earth because the earth is massive.

Experiment to determine the Universal gravitational constant G (Boy's experiment)



Two identical small gold spheres a and b of known masses m each are suspended from the ends of a highly polished bar of known length *l.* The bar is in turn suspended by a long, fine torsion wire of known torsion constant C. Two identical large lead spheres A and B of masses M each are then respectively brought near the small gold spheres a and b.
Because of the attraction between the two pairs of spheres near each other, a couple is set up, such that two equal but opposite and parallel forces F act at the ends of a polished bar.
The bar is as a result deflected through an angle θ (in radians) which can be measured by the lamp and scale method.
If d is the measured distance between the large and small spheres, then $F = \frac{GMm}{d^2}$
But, Moment of a couple = one of the forces, F × Perpendicular distance between the forces

$$= \frac{GMm}{d^2} \times l = \frac{GMml}{d^2}$$

Also resisting or opposing torque T in the torsion wire is given by; T = $c\theta$, where c is the torsional constant of the wire.
However, for the polished bar to stop rotating, then the opposing torque T produced by the torsion wire should be equal to the deflecting torque.

$$\Rightarrow \frac{GMml}{d^2} = c\theta$$

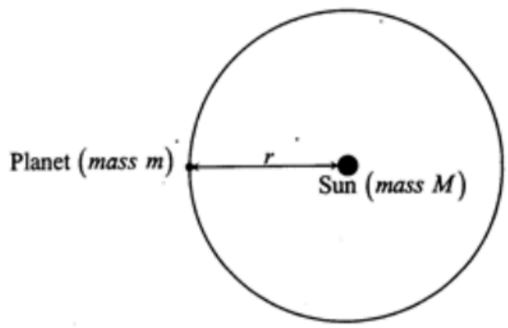
$$\therefore G = \frac{c\theta d^2}{Mml}$$

On substitution for m, M, c, d, *l,* and θ

Note:
The torsion wire should be very sensitive such that big enough deflections are obtained, which can accurately be measured by the lamp and scale method, and the whole setup of the apparatus should be small such that it can easily be screened from air convectional currents.

To show that Newton's law of gravitation is consistent with Kepler's third law
Consider a plane of mass m moving with a speed v in a circle of radius r, round the sun of mass M. Gravitational force of attraction F of the sun for the planet is given by;

$$F = \frac{GMm}{r^2}$$



Since the planet is moving in a circular path, then there should be a centripetal force for keeping it moving in a circle. This force given by $\frac{mv^2}{r}$ is provided by the gravitational attraction of the sun.

$$\Rightarrow Gravitational\ attraction = Centripetal\ force$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\therefore v^2 = \frac{GM}{r} \text{ ....(i)}$$

But $v = r\omega$ and from $T = \frac{2\pi}{\omega}$, $\omega = \frac{2\pi}{T}$

$$\Rightarrow v = \frac{2\pi r}{T}$$

$$\therefore v^2 = \frac{4\pi^2 r^2}{T^2} \text{ ....(ii)}$$

Equating equations (i) and (ii) gives;

$$\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

Since 4, π, G, M all have the same values (are constants) irrespective of the planet being considered,

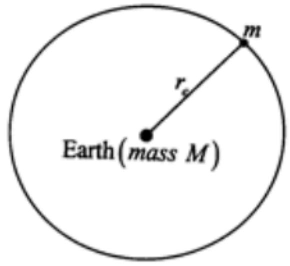
Then $\frac{4\pi^2}{GM}$ is a constant

$$\Rightarrow T^2 = kr^3$$

$\therefore T^2 \propto r^3$ which is Kepler's third law

Mass and density of Earth

Consider the Earth to have a mass M, and radius r_e, and assume that it's spherical, such that the mass is concentrated at the centre.



For a body of mass m on the Earth's surface, then there is a force of attraction equal to mg acting on it where g is the acceleration due to gravity.
From Newton's law of gravitation, the force of attraction by the earth on the body should be $\frac{GMm}{r_e^2}$
It therefore follows that the two forces are equal

$$\Rightarrow \frac{GMm}{r_e^2} = mg$$

$$\therefore M = \frac{gr_e^2}{G}$$

For g = 9.81ms^{-2} , r_e = 6400km and G = 6.67 × 10^{-11} Nm^{2}kg^{-2}

$$M = \frac{9.81 \times (6400 \times 1000)^2}{6.67 \times 10^{-11}} = 6.0 \times 10^{24} kg$$

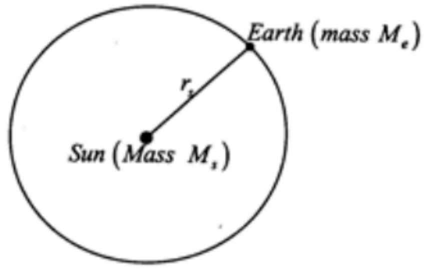
Since the earth is spherical, Volume = $\frac{4}{3}\pi r_e^3$

But $Density = \frac{Mass}{Volume} = \frac{gr_e^2}{G} \times \frac{3}{4\pi r_e^3} = \frac{3g}{4\pi G r_e}$

$$= \frac{3 \times 9.81}{4 \times \pi \times 6.67 \times 10^{-11} \times (6400 \times 1000)} = 5500\text{kgm}^{-3}$$

Mass of the sun

Consider the earth orbiting the sun, with r_s, as the radius of the earth's orbit about the sun.



As before, Centripetal force = $\frac{M_e v^2}{r_s}$ and Gravitational force = $\frac{GM_sM_e}{r_s^2}$
Equating the two equation gives;

$$\frac{M_e v^2}{r_s} = \frac{GM_sM_e}{r_s^2}$$

$$\Rightarrow v^2 = \frac{GM_s}{r_s} \quad \text{.........(i)}$$

But $v = r_s\omega$ and $\omega = \frac{2\pi}{T}$

$$\therefore v = \frac{2\pi r_s}{T} \Rightarrow v^2 = \frac{2\pi^2 r_s^2}{T^2} \quad \text{.......(ii)}$$

Thus $\frac{GM_S}{r_S} = \frac{2\pi^2 r_S{}^2}{T^2}$

$$\Rightarrow M_s = \frac{4\pi^2 r_S{}^3}{GT^2}$$

But the period of the Earth's orbit around the sun is 1 year = 365 days

$= 365 \times 24 \times 3600$ s $= 31536000$s

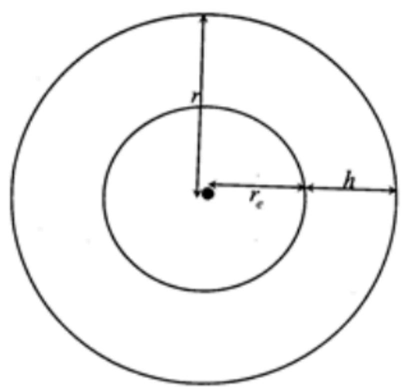
For $r_s = 1.5 \times 10^{11}$m and G = 6.67×10^{-11} Nm^{2}kg^{-2}

$$M_s = \frac{4\times\pi^2\times(1.5\times10^{11})^3}{6.67\times10^{-11}\times(31536000)^2} \approx 2 \times 10^{30}kg$$

Examples

1. A satellite is launched in a circular orbit about the equator at a height of $3.6 \times 10^4 km$ above the surface of the earth. Given that the mass of the earth is $5.98 \times 10^{24} kg$, and that its radius is 6400km, find the;
   (i) Radius of the orbit
   (ii) Speed with which the satellite is launched
   (iii) Period of the satellite

   Solution



(i) r_e = 6400km = $6.4 \times 10^6 m$

h = $3.6\times 10^4 km = 3.6 \times 10^7 m$

$r = h + r_e$

$= 3.6 \times 10^7 + 6.4 \times 10^6$

$= 4.24 \times 10^7 m$

(ii)

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{6.67\times10^{-11}\times(5.98\times10^{24})}{4.24\times10^7}} = 3067.12\text{ms}^{-1}$$

(iii)

$$v^2 = \frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\Rightarrow T^2 = \frac{4\pi^2 r^3}{GM} \qquad \therefore T = \sqrt{\frac{4\pi^2 r^3}{GM}}$$

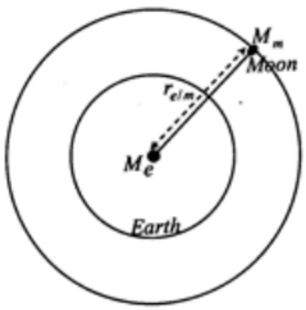
$$= \sqrt{\frac{4\times\pi^2\times(4.24\times10^7)^3}{(6.67\times10^{-11})\times(5.98\times10^{24})}}$$

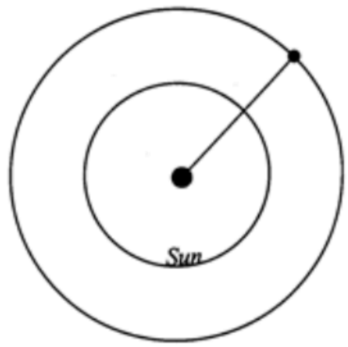
$$= 8.7\times10^4 s$$

2. Calculate the ratio of the mass of the Earth to that of the sun given that the moon moves round the Earth in a circular orbit of radius $4.0\times10^5 km$ with a period of 27.3 days and the orbital radius of the Earth round the Sun is $1.5\times10^8 km$ and its period is 365 days.

Solution

The question requires us to find $\frac{M_e}{M_S}$





$r_{e/m} = 4.0\times10^5 km = 4.0\times10^8 m$

For the earth and the moon;

$$\frac{GM_eM_m}{{r_{e/m}}^2} = \frac{M_m v^2}{r_{e/m}}$$

$$\Rightarrow v^2 = \frac{GM_e}{r_{e/m}} \text{ ....(i)}$$

But $v = r\omega$, $and\ \omega = \frac{2\pi}{r_{e/m}} \Rightarrow v^2 = \frac{4\pi^2 {r_{e/m}}^2}{{T_{e/m}}^2}$ ....(ii)

Equating the two equations gives;

$$GM_e = \frac{4\pi^2 {r_{e/m}}^3}{{T_{e/m}}^2} \text{ .... (*)}$$

For the Sun and Earth,

$$\frac{GM_SM_e}{{r_{s/e}}^2} = \frac{M_e v^2}{r_{s/e}}$$

$$\Rightarrow v^2 = \frac{GM_S}{r_{s/e}} \text{..............(iii)}$$

But $v = r\omega$, $and\ \omega = \frac{2\pi}{r_{s/e}} \Rightarrow v^2 = \frac{4\pi^2 {r_{s/e}}^2}{{T_{s/e}}^2}$ ....(iv)

Equating the two equations gives;

$$GM_s = \frac{4\pi^2 {r_{s/e}}^3}{{T_{s/e}}^2} \text{...................(**)}$$

Dividing equation (*) by equation (**) gives;

$$\frac{GM_e}{GM_S} = \frac{4\pi^2 {r_{e/m}}^3}{{T_{e/m}}^2} \times \frac{{T_{s/e}}^2}{4\pi^2 {r_{s/e}}^3}$$

$$\Rightarrow \frac{M_e}{M_s} = \frac{r_{e/m}{}^3 \times T_{s/e}{}^2}{T_{e/m}{}^2 \times r_{s/e}{}^3}$$

$$T_{e/m} = 27.3\ days = 27.3 \times 24 \times 60 \times 60 = 2.4 \times 10^6 s$$

$$T_{s/e} = 365\ days = 365 \times 24 \times 60 \times 60 = 3.2 \times 10^7 s$$

$$\Rightarrow \frac{M_e}{M_s} = \frac{(4.0\times10^8)^3\times(3.2\times10^7)^2}{(2.4\times10^6)^2\times(1.5\times10^{11})^3} = 3.4 \times 10^{-6}$$

Trial questions

1. When a space shuttle is in an orbit at a mean height of $0.33 \times 10^6 m$ above the surface of the earth, it requires 91 minutes to complete one orbit. Find the value of the mass of the Earth. [Ans: $6.0\times 10^{24} kg$]
2. The Earth of mass M_e describes a circular orbit of radius $r_{s/e}$ about the sun of mass M_s and radius r_s at an angular velocity 2.0×10^{-7}rad s^{-1} due to gravitational attraction. Given that $\frac{r_{s/e}}{r_s} = 200$, calculate the density of the sun. [Ans: 1100kgm^{-3}]
3. The distance of Jupiter from the sun is 5.2 times the distance of the earth from the sun. Assuming that both orbits are circular, find the time Jupiter takes to complete its orbit, if the earth takes one year.
4. Two binary stars of masses m_1 and m_2 respectively rotate about their common centre of mass with an angular speed ω. Assuming that the only force acting on the stars is the mutual gravitational force between them, and the distance between them is d, show that:

$$\omega = \frac{[G(m_1+m_2)]^{\frac{1}{2}}}{d^{\frac{3}{2}}}$$, where G is the universal gravitation constant

ACCELERATION DUE TO GRAVITY

- **Relationship between g and G**

A body of mass m on the earth's surface experiences a force equal to mg. If M is the mass of the Earth, which is assumed to be concentrated at its centre, the gravitational attraction by the Earth on the body is; $\frac{GM_e m}{r_e{}^2}$

$$\therefore \frac{GM_e m}{r_e{}^2} = mg \ \Rightarrow g = \frac{GM_e}{r_e{}^2} \ldots\ldots\ldots (i)$$

- **Variation of g with height**

For a body of mass m at a distance $r > r_e$ from the centre of the Earth, if the acceleration due to gravity at that point is g', then;

$$\frac{GMm}{r^2} = mg' \ \Rightarrow g' = \frac{GM}{r^2} \ldots\ldots\ldots (ii)$$

Dividing equation (ii) by equation (i) gives:

$$\frac{g'}{g} = \frac{GM}{r^2} \times \frac{r_e{}^2}{GM}$$

$$\frac{g'}{g} = \frac{r_e{}^2}{r^2} \Rightarrow g' = \frac{r_e{}^2}{r^2} g$$

Since r_e^2 *and* g are constants, $g' \propto \frac{1}{r^2}$

Therefore, for points above or outside the earth's surface, the acceleration due to gravity varies as inversely as the square of the distance r from the centre of the Earth.

On the other hand, for a body of mass m which is inside the earth, i.e at a distance $r < r_e$ from the centre of the earth. If the acceleration due to gravity at that point is g', then;

$$\frac{GMm}{r^2} = mg' \quad \Rightarrow M = \frac{r^2 g'}{G} \text{ and from equation (i), } M_e = \frac{r_e^2 g}{G}$$

If the Earth is assumed to have uniform density, ρ, *then* ; $M = \frac{4}{3}\pi r_e^3 \rho$

Therefore, $\frac{M}{M_e} = \frac{r^3}{r_e^3}$

Substituting for Me and M:

$$\frac{r^2 g'}{G} \times \frac{G}{r_e^2 g} = \frac{r^3}{r_e^3} \Rightarrow g' = \frac{r}{r_e} g$$

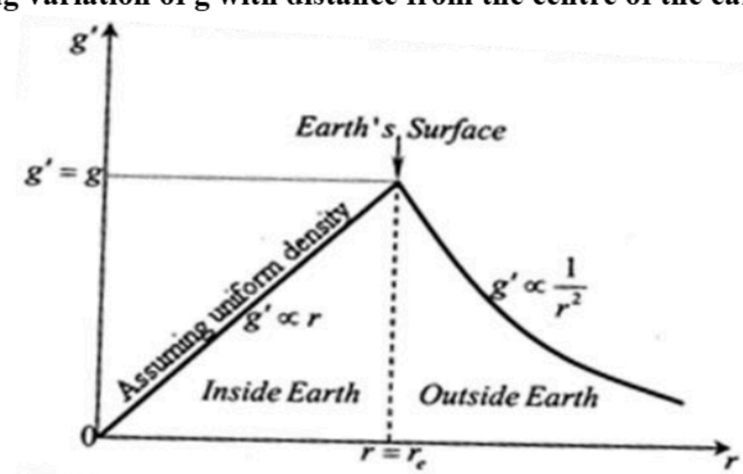
It can therefore be seen for values of $r < r_e$, then the acceleration due gravity varies directly with the distance from the centre.

The reader should however note that throughout the derivations, it has been assumed that the density of the Earth is uniform, which is not actually true. However, it is normal practice to make this assumption at this level

It can therefore be summarized that;

- Inside the earth surface i.e for $r < r_e$, then acceleration due to gravity , $g' \propto r$
- Outside the earth surface i.e for $r > r_e$, then acceleration due to gravity, $g' \propto \frac{1}{r^2}$

Graph showing variation of g with distance from the centre of the earth



From the graph, and the equations obtained above, it can be seen that the acceleration due to gravity reduces as one moves away from the Earth's surface. This explains why the diffusion of gases is easier in the outer space than at places near the Earth's surface. It's however also due to little atmosphere in the outer space.

Note:

The force acting on a body of mass m at a distance r from the centre of the earth is given by;

$F = mg' \qquad \Rightarrow g' = \frac{F}{m}$

g' is called the gravitational intensity or gravitational field strength.

The SI unit of g' is Nkg^{-1} or ms^{-2}

Gravitational intensity can be defined as the force acting on a mass of 1kg placed in the Earth's gravitational field.

OR

It is the force of attraction of a planet on a mass of 1kg

Examples

1. At what distance from the Earth's surface will the acceleration be $\frac{1}{8}$ of its value at the earth's surface?

Solution

On the Earth's surface, $mg = \frac{GM_e m}{r_e^2} \Rightarrow g = \frac{GM_e}{r_e^2}$ …………..(i)

Away from the Earth's surface, $mg' = \frac{GM_e m}{r^2} \Rightarrow g' = \frac{GM_e}{r^2}$ ………(ii)

Dividing the two equation gives:

$$\frac{G}{g'} = \frac{GM_e}{r_e^2} \times \frac{r^2}{GM_e} = \frac{r^2}{r_e^2} \quad \text{but } g' = \frac{1}{8}g$$

$$\Rightarrow \frac{g}{\frac{1}{8}g} = \frac{r^2}{r_e^2} \qquad \therefore r = \sqrt{8r_e^2} \quad \text{for} \quad r_e = 6.4 \times 10^6$$

$$r = \sqrt{8 \times (6.4 \times 10^6)^2} = 1.81 \times 10^7 m$$

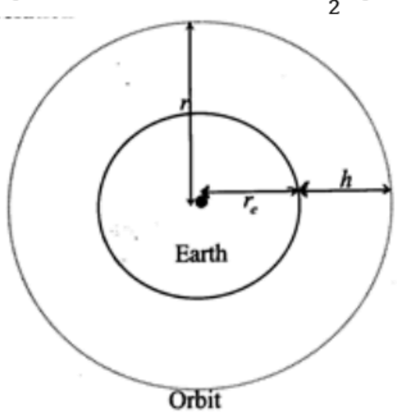
r is the distance from the centre of the earth, and so we have to subtract the radius of the earth from this value to get the distance from the Earth's surface.

Required distance, $h = r - r_e = 1.81 \times 10^7 - 6.4 \times 10^6 = 11.7 \times 10^6 m$

2. A body weighs 63N on the Earth's surface. How much will it weigh at a height equal to half the radius of the Earth, above the Earth's surface?

Solution

Radius of Earth, $r_e = 6.4 \times 10^6 m \Rightarrow h = \frac{1}{2} r_e = 3.2 \times 10^6 m$



$r = r_e + h = 9.6 \times 10^6 m$

W = mg

For W = 63N, $m = \frac{W}{g} = \frac{63}{9.81} = 6.422kg$

Also for a body placed at a point from the centre of the Earth, $\frac{GM_e m}{{r_e}^2} = mg$

$$\Rightarrow GM_e = g{r_e}^2 = 9.81 \times (6.4 \times 10^6)^2$$
$$GM_e = 4.02 \times 10^{14} \ldots\ldots\ldots (i)$$

When the body is at a distance *r* from the centre of the Earth, $\frac{GM_e m}{r^2} = mg'$ where g' is the acceleration due to gravity at that point.

$$\Rightarrow GM_e = g'r^2 = g'(9.6 \times 10^6)^2$$
$$GM_e = (9.6 \times 10^6)^2\, g' \ldots\ldots\ldots\text{(ii)}$$

Equating equations (i) and (ii) gives;

$$(9.6 \times 10^6)^2\, g' = 4.02 \times 10^{14}$$
$$\Rightarrow g' = \frac{4.02\times10^{14}}{(9.6\times10^6)^2} = 4.36\text{ms}^{-2}$$

Weight of the body W' at that point = $mg' = 6.422 \times 4.36 = 28N$

Variation of g with latitude

The acceleration due to gravity varies slightly from one place to another on the Earth's surface. For example, its value for a body at the pole is different from what it is when the body is at the equator, its value at the poles being slightly higher. This variation is due to:

- The Earth is not a perfect sphere. Therefore, all places on the Earth's surfaces are not at the same distance from the centre of the Earth. For example, a body at the equator is slightly further away from the centre of the earth compared to one at the poles, and hence feels a smaller gravitational acceleration. The nearer the place with respect to the centre of the earth, the greater the value of g.

- The acceleration due to gravity also varies due to the effect of the Earth's rotation.

The Earth rotates about its axis once in 24 hours. A body on the surface of the also rotates with it in a circular path. Since the body is in a rotating frame, it experiences an outward centrifugal force against the inward force of gravity. The effect of the Earth's rotation therefore reduces the acceleration due to gravity. At the poles, the radius of the circular path is zero, and so there so there is no effect of the Earth's rotation on the value of acceleration due to gravity. However at the equator, the radius of the circular path is maximum, and so the effect of the Earth's rotation on the value of the acceleration due to gravity is maximum.

Parking (Geostationary) Orbits

A parking orbit is a path in space of a satellite which makes it appear to be in the same position relative to the observer at a point on the Earth.

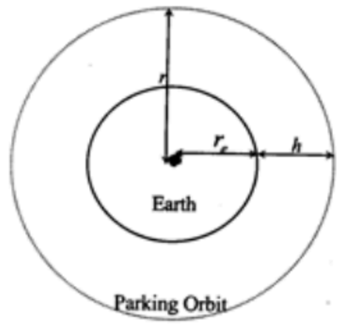
The period of the satellite should be equal to the period of the Earth (=24 hrs), and the satellite should be moving in the same direction as the Earth is rotating.

A satellite in a parking orbit is also referred to as a geosynchronous satellite

Example

Calculate the radius of a parking orbit for an Earth satellite, and find the height of this orbit above the Earth's surface.

Solution



For circular motion, centripetal force = $\frac{mv^2}{r}$

Gravitational attraction of the planet by the earth = $\frac{GM_e m}{r^2}$

$$\Rightarrow \frac{mv^2}{r} = \frac{GM_e m}{r^2} \quad \therefore v^2 = \frac{GM_e}{r} \text{...........(i)}$$

Also, from $= r\omega,\ and \quad \omega = \frac{2\pi}{T}$,

$$\Rightarrow v = \frac{2\pi r}{T} \quad \therefore v^2 = \frac{4\pi^2 r^2}{T^2} \text{.........(ii)}$$

Equating two equations: $\frac{GM_e}{r} = \frac{4\pi^2 r^2}{T^2}$

$$\Rightarrow T^2 = \frac{4\pi^2 r^3}{GM_e} \text{ but } GM_e = g{r_e}^2$$

$$\therefore T^2 = \frac{4\pi^2 r^3}{g{r_e}^2}$$

For a parking orbit for the Earth, its period should be equal to the period of the Earth, which is 24 hours. $T = 3600 \times 24 = 86400s$

$$\therefore (86400)^2 = \frac{4\times\pi^2\times r^3}{9.81\times(6.4\times10^6)^2}$$

$$\Rightarrow r = \sqrt[3]{\frac{(86400)^2\times9.81\times(6.4\times10^6)^2}{4\times\pi^2}} = 4.24\times10^7 m$$

Radius of orbit $= 4.24 \times 10^7 m$

$$h = r - r_e = 4.24\times10^7 - 6.4\times10^6 = 3.6 \times 10^7 m$$

Trial question

A communication satellite is placed in an orbit such that it remains directly above a fixed point on the earth's surface at all times.

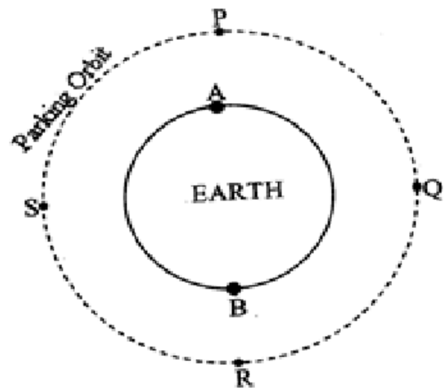
(i) What is the period of this satellite?

(ii) Explain why the satellite must be in orbit above the equator

(iii) Show that the correct height for the orbits does not depend on the mass of the satellite.

Uses of the parking orbits

Parking orbits are used in worldwide telecommunications



A set of three or more satellites are launched in a parking orbit as shown in the figure. Assuming that it required that radio signals be transmitted from A to B, the signals can be transmitted from A to a geosynchronous satellite P, then re-transmitted from P to another geosynchronous satellite Q, then to R, and finally to B.

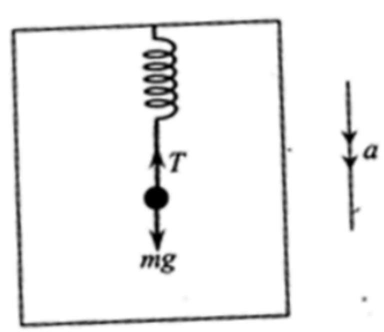
The signals could also take the paths;

$A \rightarrow P \rightarrow S \rightarrow R \rightarrow B$

It should however be noted that the use of satellites in communication by putting them in a parking orbit is only possible when the satellites' orbit are in the equatorial plane, and that only 'line of sight communication' is possible on satellite communications, implying that communication can only occur provided that there is no obstruction between the transmitter, the satellite and the receiver. This is the disadvantage in communication using geostationary satellites.

Weightlessness

- **Case 1:**



Consider a body of mass m hanging from a spring balance which is suspended from the roof of a lift. If the lift moves down with an acceleration of a ms-2, then;

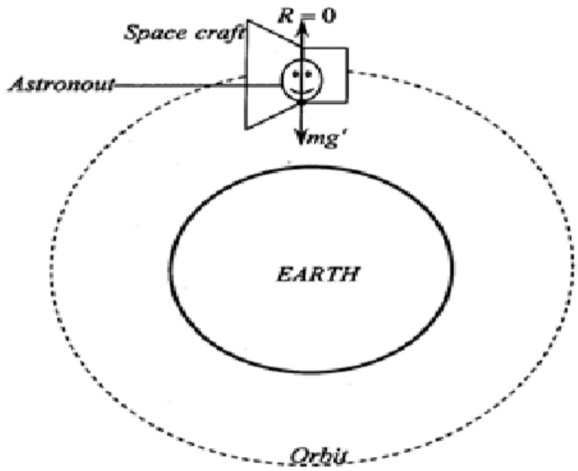
$mg - T = ma$. If the lift is moving with a constant velocity, then $a = 0$, and so $mg = T$, implying that the spring balance gives a reading of the weight of the body.

However, if the lift falls freely such that its downward acceleration is a is equal to the acceleration a is equal to the acceleration due to gravity g, then; $mg - T = mg \Rightarrow T = 0$

It therefore follows that the spring balance would register the mass of that the spring balance would register the mass of the body as being zero, and so the body is said to be weightless

- **Case 2:**

Consider an astronaut in a space craft which is orbiting the Earth as shown in the figure.



At a particular height of the orbit, its possible for the space craft to move such that its centripetal acceleration is equal to the acceleration due to gravity at that height.

$$\Rightarrow a = g'$$

$$\therefore mg' - R = mg' \Rightarrow R = 0$$

The astronaut therefore experiences no reaction from the space craft, and so becomes weightless.

Therefore **weightlessness** can be defined as the condition of zero reaction experienced by an astronaut in accelerating space craft.

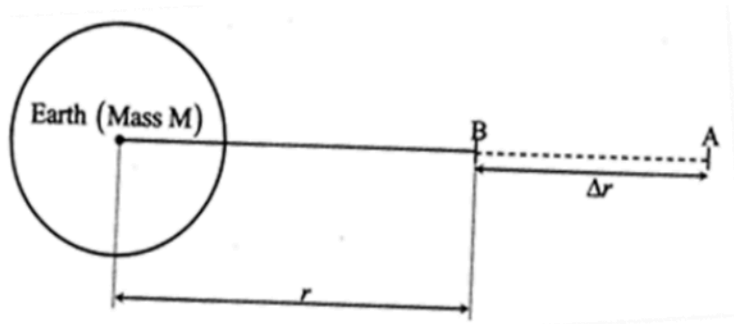
Gravitational potential

Gravitational potential at a point due to the gravitational field of the Earth is the work done in moving a body of mass 1kg from infinity to that point.

$\therefore From\ F = \frac{GMm}{r^2}$, if m = 1kg,

Then $F = \frac{GM\times 1}{r^2} = \frac{GM}{r^2}$

Consider a body of mass 1kg being moved from A to B through a small distance Δr, where B is at a distance r from the centre of the earth as shown below.



Work done in moving the body through a distance Δr is given by; $\Delta w = F\Delta r$

$$\Rightarrow dw = Fdr$$

But $F = \frac{GM}{r^2} \quad \Rightarrow dw = \frac{GM}{r^2}dr$

Therefore, the total work done in moving the 1kg mass from infinity to some point a distance r from the centre of the earth in its gravitation field can be obtained by summing the small works done from infinity to that point.

$$\therefore \int dw = \int_{\infty}^{r} \frac{GM}{r^2} dr$$

$$\Rightarrow W = GM\int_{\infty}^{r} r^{-2}dr$$

$$= GM\left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$= GM\left[-\left(\frac{1}{r} - \frac{1}{\infty}\right)\right] = GM\left(-\frac{1}{r}\right)$$

Therefore gravitational potential = $-\frac{GM}{r}$

On the earth's surface, $r = r_e$, $\Rightarrow$ Gravitational potential = $-\frac{GM}{r_e}$

It should be noted that if a body being moved from infinity to a point close to the earth's surface is not of mass 1kg, then the work done is not gravitational potential, but is called gravitational potential energy or simply potential energy.

Therefore potential energy = $-\frac{GMm}{r}$

On the earth's surface, potential energy $= -\frac{GMm}{r_e}$

Gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done in bringing the mass m from infinity to a distance r

$$dw = Fdr, \quad w = \int_{\infty}^{r} \frac{GMm}{r^2} dr = GMm\int_{\infty}^{r} r^{-2}dr = GMm\left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$\therefore PE = -\frac{GMm}{r}$$

The negative sign implies that the potential energy at infinity is zero and greater than the potential energy at points close to the Earth's surface. The decrease in P.E as the mass m moves in from infinity towards the mass M implies that the work done by the gravitational force of mass M on a mass m is positive. This follows from the work energy theorem.

However the amount of work done against the gravitational force of mass M to move the mass m from a distance r_1 *to position* r_2 *is given by;*

$$w = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = change\ in\ potential\ energy$$

ESCAPE VELOCITY

Consider a rocket of mass m fired from the Earth's surface such that it escapes from gravitational influence of the earth. The rocket requires a particular velocity, and hence a particular amount of energy to escape. The velocity that the rocket should have is got from the fact that;

Kinetic energy lost = Potential energy gained

$$\Rightarrow \frac{1}{2}mu^2 = 0 - -\frac{GMm}{r_e}$$

$$\therefore u = \sqrt{\frac{2GM}{r_e}} \quad but\ GM = gr_e^{\,2}$$

$$\Rightarrow u = \sqrt{\frac{2gr_e^{\,2}}{r_e}} = \sqrt{2gr_e}$$

If $r_e = 6.4 \times 10^6 m$ and $g = 9.81 ms^{-2}$

$$u = \sqrt{2 \times 9.81 \times 6.4 \times 10^6} \approx 11 kms^{-1}$$

For other planets, the escape velocity, $v = \sqrt{\frac{2GM}{r'}} = \sqrt{2g'r'}$, *where* g' *is* the acceleration due to gravity for the planet and r' the radius of the planet.

For example the escape velocity on the surface of the moon of radius $1.72 \times 10^6 m$ and mass $7.36 \times 10^{22} kg$ can be found as below,

$$v = \sqrt{\frac{2GM}{r'}} = \sqrt{\frac{2\times(6.67\times10^{-11})\times(7.36\times10^{22})}{1.72\times10^6}} = 2.39 \times 10^3 ms^{-1} = 2.39 kms^{-1}$$

Definition:

Escape velocity is the vertical velocity that a body should be given at the surface of a planet (such as Earth) to so that it just escapes from the gravitation attraction of the planet.

OR

It is the minimum velocity with which a body is projected in order that it may escape the earth's gravitational pull.

It is after knowing the concept of escape velocity and the effect of of mass on the gravitational force of attraction between bodies that we are able to explain why the moon has no atmosphere, yet the sun has atmosphere

Because of its small mass, the gravitational pull of the moon is too weak to hold lighter gases such as helium and hydrogen. These gases therefore have average molecular velocities of order of escape velocity of the moon (which is ≈ $2.5 kms^{-1}$). The gases therefore escape and the moon remains with no atmosphere. It is for the same reason that the moon is cold.

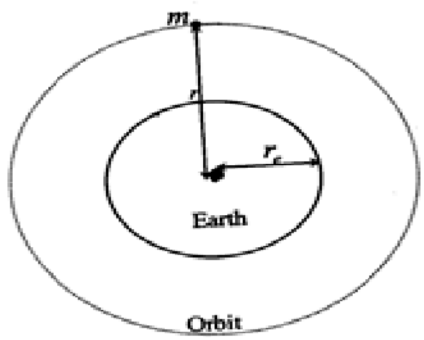
However, because the sun has a large mass, its gravitational pull is stronger, and it has a much higher escape velocity (≈ $620 kms^{-1}$) and so gases can never have enough velocities to escape.

Trial question

The gravitational potential E at the surface of a planet of mass M and radius R is given by $E = -\frac{GM}{R}$, where G is the universal gravitation constant. Derive an expression for the lowest velocity, v , which an object of mass m must have at the surface of the planet if it is to escape from the planet. [Ans: $v = \left(\frac{2GM}{R}\right)^{\frac{1}{2}}$]

ENERGY OF A SATELLITE

Consider a satellite of mass m launched from the Earth's surface of radius *re* into an orbit of radius *r* as shown in the figure.



While in orbit, the satellite has both potential and kinetic energy.

Potential energy on the earth's surface = $\frac{GM_em}{r_e}$ …………(i)

Centripetal force on the body = $\frac{mv^2}{r}$ and Gravitational force on the satellite = $\frac{GM_em}{r^2}$

$$\therefore \frac{mv^2}{r} = \frac{GM_em}{r^2} \Rightarrow mv^2 = \frac{GM_em}{r}$$

$dividing\ through\ by\ 2\ gives;$

$$\frac{1}{2}mv^2 = \frac{GM_em}{2r} = kinetic\ energy$$

$$\therefore kinetic\ energy = \frac{GM_em}{2r} \dots\dots\dots\dots\dots(ii)$$

From equations (i) and (ii), it can be seen that the potential energy is twice the kinetic energy but of opposite sign

Also for the satellite in orbit,

Mechanical energy = Gain in potential energy + kinetic energy

$$= \left(-\frac{-GM_em}{r} - \frac{-GM_em}{r_e}\right) + \frac{GM_em}{2r}$$

$$= \frac{GM_em}{r_e} - \frac{-GM_em}{r} + \frac{GM_em}{2r} = \frac{GM_em}{r_e} - \frac{GM_em}{2r} = GM_em\left(\frac{1}{r_e} - \frac{1}{2r}\right)$$

Since $GM = gr_e^2, total\ energy = m\,gr_e^2\left(\frac{1}{r_e} - \frac{1}{2r}\right)$

However, if a body or satellite is not launched from the earth's surface;

Then; mechanical energy = potential energy + kinetic energy

$$= \frac{-GM_e m}{r} + \frac{GM_e m}{2r}$$

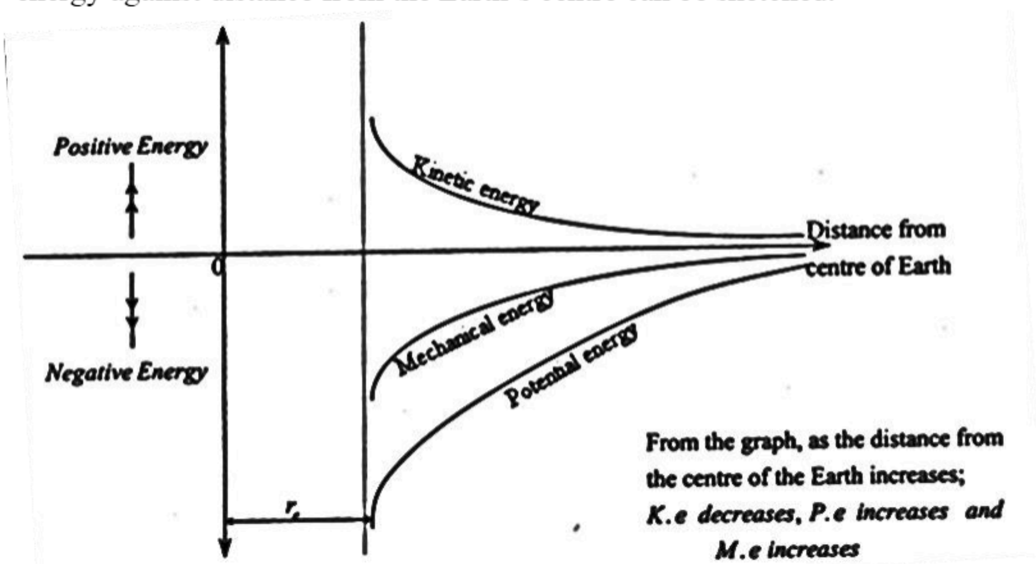
$$= \frac{GM_e m}{r}\left(-1 + \frac{1}{2}\right) = \frac{-GM_e m}{2r}$$

Since $GM_e = gr_e{}^2, Mechanical\ energy\ (total\ enegy) = \frac{-mgr_e{}^2}{2r}$

It should be noted, that due to friction of the Earth's atmosphere, the radius of the orbit reduces and from P.e = $\frac{-GM_e m}{r}$, it follows that the potential energy reduces as well. Also from K.e = $\frac{GM_e m}{2r}$, kinetic energy increases, and since K.e = $\frac{1}{2}mv^2$, it follows that velocity increases as well.

Also, since potential energy is twice kinetic energy, and potential energy has reduced, Mechanical energy also reduces, and if no precautions are taken, the satellite may burn due to the heat produced by friction.

From the expressions for kinetic energy, potential energy and mechanical energy, a graph of energy against distance from the Earth's centre can be sketched.

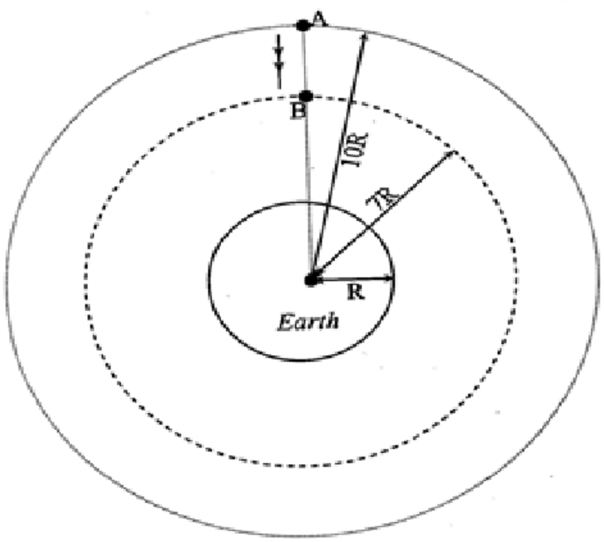


Examples

1. A mass is released from a point at a distance of 10R from the centre of the Earth, where R is the radius of the Earth. Find the speed of the mass at a point a distance of 7R from the centre of the Earth

   Solution

   Consider the body to have been released from A, and that when at B, it's a distance of 7R from the centre of the Earth.



At A:

Since it was released, u = 0 $\Rightarrow k.e = 0$

P.e = $\frac{-GM_em}{r} = \frac{-GM_em}{10R}$

Mechanical energy at A = P.e + k.e

$= \frac{-GM_em}{10R} + 0 = \frac{-GM_em}{10R}$

At B:

Kinetic energy = $\frac{GM_em}{2r} = \frac{1}{2}mv^2$

Potential energy = $\frac{-GM_em}{r} = \frac{-GM_em}{7R}$

$\Rightarrow$ Mechanical energy at B = $\frac{1}{2}mv^2 + \frac{-GM_em}{7R}$

Basing on an assumption that mechanical energy is conserved, then mechanical energy at A should be equal to the mechanical energy at B.

$\Rightarrow \frac{1}{2}mv^2 + \frac{-GM_em}{7R} = \frac{-GM_em}{10R}$

$\frac{1}{2}v^2 = \frac{-GM_e}{10R} + \frac{GM_e}{7R}$

$\frac{1}{2}v^2 = \frac{10GM_e - 7GM_e}{70R} = \frac{3GM_e}{70R}$

But $GM_e = gr_e{}^2 = gR^2$

$\Rightarrow v^2 = \frac{6gR^2}{70R}$ $\quad \therefore v = \sqrt{\frac{6gR}{70}}$

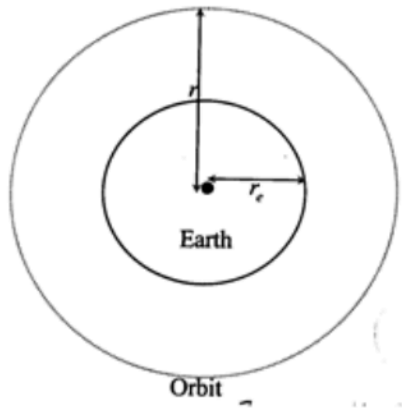
But g = 9.81 and R = $6.4 \times 10^6 m \Rightarrow v = \sqrt{\frac{6 \times 9.81 \times 6.4 \times 10^6}{70}}$ = 2.3×10^3 ms^{-1}

2. A satellite of mass 100kg is launched in a circular orbit at a height of 12800km about the surface of the earth.
   (i) Find the period of the satellite
   (ii) Calculate the mechanical energy of the satellite

(iii) What would happen if the satellite's speed was halved while it's in orbit?

Solution

(i) $r_e = 6.4 \times 10^6 m \ and \ r = 1.28 \times 10^8 m$



$$\frac{mv^2}{r} = \frac{GM_e m}{r^2} \quad \Rightarrow v^2 = \frac{GM_e}{r}$$

$$But\ v^2 = \frac{4\pi^2 r^2}{T^2}$$

$$\Rightarrow \frac{GM_e}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$T^2 = \frac{4\pi^2 r^3}{GM_e} \text{ but } GM_e = gr_e^2$$

$$\therefore T = \sqrt{\frac{4\pi^2 r^3}{gr_e^2}} = \sqrt{\frac{4\times\pi^2\times(1.28\times10^8)^3}{9.81\times(6.4\times10^6)^2}}$$

$$= 4.5 \times 10^5 s$$

(ii)

Mechanical energy = Potential energy + Kinetic energy

$$= \frac{-GM_e m}{r} + \frac{GM_e m}{2r} = \frac{-GM_e m}{2r}$$

But $GM_e = gr_e^2 \Rightarrow$ Mechanical energy $= \frac{-gr_e^2 m}{2r}$

$$= \frac{9.81\times(6.4\times10^6)^2}{2\times(1.28\times10^8)} = 1.57 \times 10^8 \text{J}$$

(iii)

Initially, $\frac{GM_e m}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM_e}{r}$ ……….(i)

When the speed is halved, $\frac{GM_e m}{r_0^2} = \frac{m\left(\frac{1}{2}v\right)^2}{r_0} \Rightarrow v^2 = \frac{4GM_e}{r_0}$ …………(ii)

Equating equations (i) and (ii) gives;

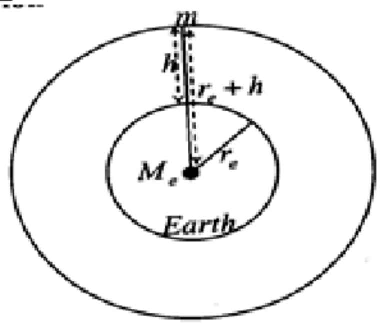
$$\frac{GM_e}{r} = \frac{4GM_e}{r_0}$$

$$\therefore r_0 = 4r$$

Therefore, the radius of the new orbit is four times the radius of the original orbit, and from k.e $= \frac{GM_e m}{2r}$, it follows that k.e reduces, from P.e $= \frac{-GM_e m}{r}$, potential energy increases and hence mechanical energy increases

3. A rocket is launched from the Earth's surface with a velocity v. If h is the maximum height above the earth's surface it would go, show that: $h = \frac{v^2 r_e}{2gr_e - v^2}$, where re is the radius of the earth.

Solution



While on the earth's surface;

K.e = $\frac{1}{2}mv^2$ $\quad$ *and* $P.e = \frac{-GM_e m}{r_e}$

$\Rightarrow$ m.e $= \frac{1}{2}mv^2 - \frac{GM_e m}{r_e}$

At maximum height;

Speed is zero, $\Rightarrow$ k.e $= \frac{1}{2}m \times 0^2 = 0$ and $\quad$ p.e $= \frac{-GM_e m}{r_e + h}$

$\therefore$ m.e $= \frac{-GM_e m}{r_e + h}$

From the principle of conservation of mechanical energy, mechanical energy at the Earth's surface should be equal to the mechanical energy at the maximum point reached.

$\Rightarrow \frac{1}{2}mv^2 = 0 - \frac{GM_e m}{r_e+h} = \frac{-GM_e m}{r_e+h} \therefore \frac{mv^2 r_e - 2GM_e m}{2r_e} = \frac{-GM_e m}{r_e+h}$

But $GM_e = gr_e^2$,

Substituting for GM_e, and dividing through by m gives: $\frac{v^2 - 2gr_e}{2} = \frac{-gr_e^2}{r_e+h}$

Cross multiplying gives: $v^2 r_e + v^2 h - 2gr_e^2 - 2gr_e h = -2gr_e^2 \Rightarrow v^2 r_e = 2gr_e h - v^2 h$

$\therefore \; h(2gr_e - v^2) = v^2 r_e \Rightarrow h = \frac{v^2 r_e}{2gr_e - v^2}$

Trial questions

1. A satellite of mass 66kg is in orbit round the earth at a distance of 5.7R above the earth's surface, where R is the radius of the earth. If the gravitational field strength at the earth's surface is 9.8 Nkg^{-1}, calculate the centripetal force acting on the satellite. Assuming that the earth's radius is 6400km, calculate the period of the satellite in orbit. [Ans. F = 14.4 N, T = 2116800s]

2. A radio astronomy research satellite of mass 200kg circles the earth in an orbit of radius $\frac{3R}{2}$, where R is the radius of the earth. If the gravitational pull on a mass of 1kg at the earth's surface is 10N, calculate the pull on the satellite.

CHAPTER 12: SIMPLE HARMONIC MOTION

Any motion that repeats itself after a certain period of time is called a periodic motion, and if such a motion can be represented in terms of sines and cosines, then it is called harmonic motion.

Definition: Simple harmonic motion (s.h.m) is a periodic motion whose acceleration is directly proportional to the displacement from a fixed point, and it is directed towards that point.

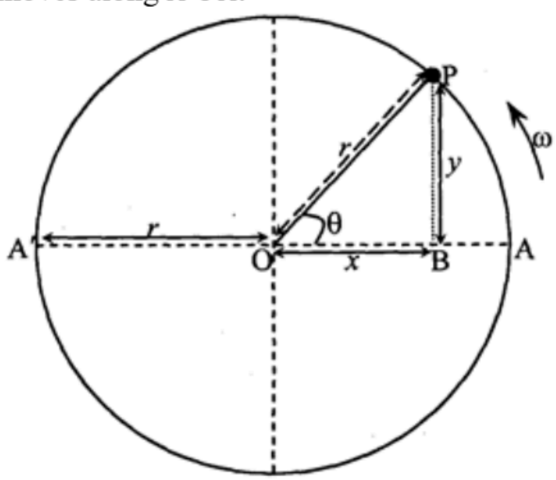
Characteristics of s.h.m

- It is periodic
- Its acceleration is directly proportional to the displacement from a fixed point
- Its acceleration is always directed towards directed towards a fixed point in the line of motion,
- Mechanical energy is always conserved

Practical examples of s.h.m

- Pendulum bob
- Balance wheel of a watch
- Pistons in a petrol engine,
- Strings in musical instruments

Consider a body P moving in a circular path of radius r with a constant angular velocity of ω, as shown. Let the perpendicular projection of P onto the diameter of the circle be B. As P moves along the circle, B moves along $A'OA$.



r is the maximum displacement from the rest position, and is called the amplitude

From the figure; $x = r\cos\theta$ and $y = r\sin\theta$

But $\omega = \frac{\theta}{t}$ $\Rightarrow \theta = \omega t$

$\therefore x = r\cos\omega t$ and $y = r\sin\omega t$

For $r = A = amplitude,$ $x = A\cos\omega t$ and $y = A\sin\omega t$

$$v = \frac{dx}{dt} = \frac{d}{dt}(A\cos\omega t) = -A\omega\sin\omega t$$

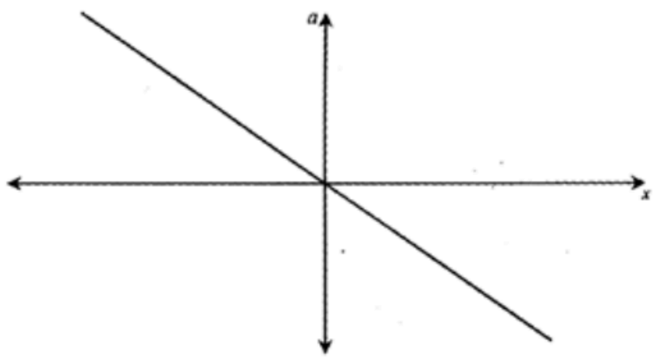
$$a = \frac{dv}{dt} = \frac{d}{dt}(-A\omega \sin \omega t) = -\omega^2(A\cos \omega t)$$

$But\ A\cos \omega t = x \qquad \Rightarrow a = -\omega^2 x$

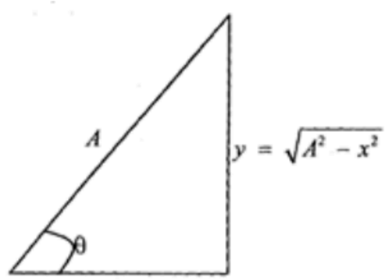
Since $-\omega^2$ is a constant, then $a \propto x$

Therefore, if any motion can be shown to obey $a = -\omega^2 x$, then that motion is simple harmonic. The negative sign on the right hand side of the equation means that the acceleration is directed towards the equilibrium position, for if it were not there, it would imply that the body goes on accelerating and will never return to the equilibrium position.

Graph of *a against x for s.h.m*



Also from $v = -A\omega \sin \omega t$



But $\sin\theta = \sin \omega t = \frac{\sqrt{A^2-x^2}}{A} \qquad \therefore v = -\omega\sqrt{A^2 - x^2}$

Since the square root of a number always results in a positive and negative value, then;

$$v = \pm\omega\sqrt{A^2 - x^2}$$

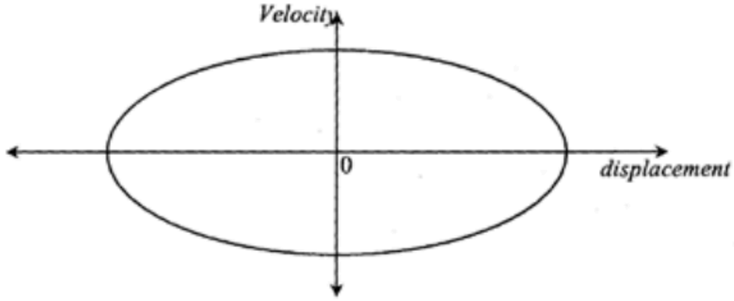
It can be noted from the expression for velocity that velocity is maximum when $x = 0$. This is at the centre of oscillation

$$V_{max} = A\omega$$

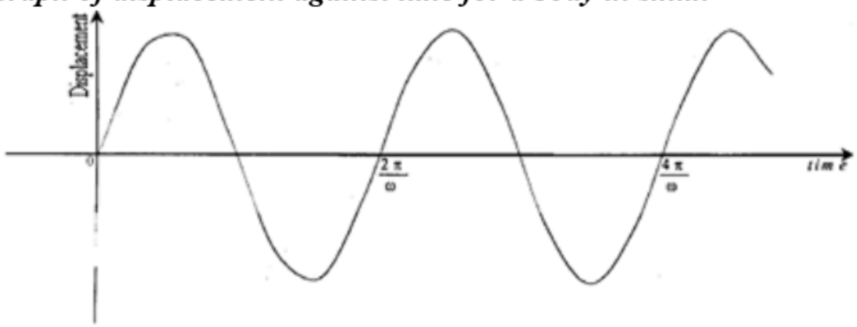
It can also be seen that velocity is zero when $x = A$, *i.e* at the point of maximum displacement, where the particle or body is momentarily at rest.

It should therefore follow that acceleration is zero at the centre of oscillation, i.e when $x = 0$, *and* maximum *when* $x = A \qquad \therefore a_{max} = |-\omega^2 A|$

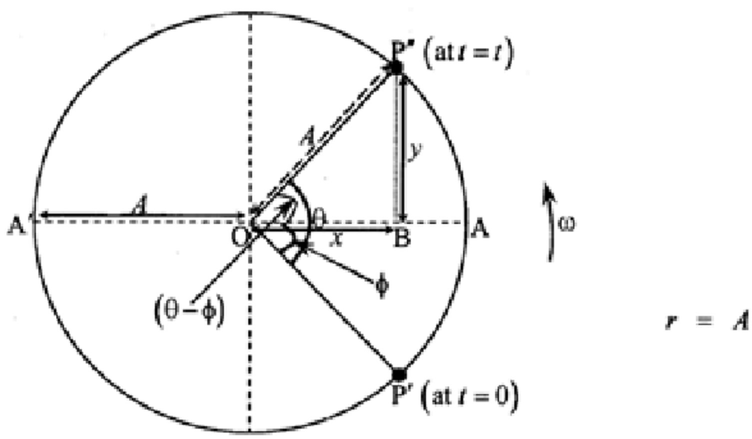
Graph of velocity against displacement for a body in s.h.m



Graph of displacement against time for a body in s.h.m



Case 1:

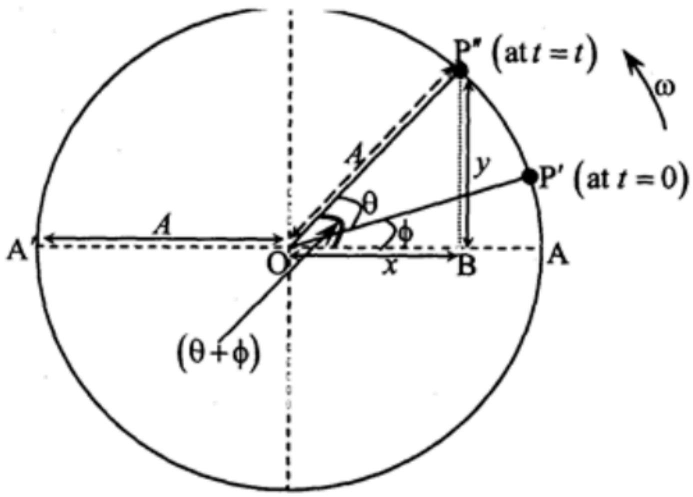


Consider a particle starting from point P' at a time $t = 0$, and after a time t, it is at position P''

$$\angle P''OA = (\theta - \phi) \text{ and since } \theta = \omega t, \quad \angle P''OA = \omega t - \phi$$

$$\therefore x = A\cos(\omega t - \phi) \ \ and\ y = A\sin(\omega t - \phi)$$

Case 2:



Also , if a particle starting from point P' at a time $t = 0$, and after a time t, its position P''.

$$\angle P''OA = (\theta + \phi) \text{ and since } \theta = \omega t, \quad \angle P''OA = \omega t + \phi$$

$$\therefore x = A\cos(\omega t + \phi) \text{ and } y = A\sin(\omega t + \phi)$$

For the two cases above, the angle θ is called the phase angle/phase difference/epoch. This angle depends on the position of the particle commences.

Examples

1. Given that $x = 8\cos\left(0.5\pi t + \frac{\pi}{3}\right)$, describes the displacement of a particle from rest position, find the;
   (i) Amplitude,
   (ii) Frequency
   (iii) Period
   (iv) Maximum velocity
   (v) Maximum acceleration
   (vi) Phase angle

Solution

Comparing the given equation $x = 8\cos\left(0.5\pi t + \frac{\pi}{3}\right) m$ with an already known equation $x = r\cos(\omega t + \phi)$, shows that;

(i) *Amplitude r = 2m*

(ii) $\omega t = 0.5\pi t \quad \Rightarrow \omega = 0.5\pi$

and since $frequency = \frac{1}{T} = \frac{\omega}{2\pi}$

frequency $= \frac{0.5\pi}{2\pi} = 0.25\ Hz$

(iii) Period T $= \frac{1}{frequency} = \frac{1}{0.25} = 4s$

(iv) $V_{max} = A\omega = 0.5\pi \times 8 = 4\pi \; ms^{-1}$

(v) $a_{max} = |-\omega^2 x|$ but for maximum acceleration, $x = A$

$\Rightarrow a_{max} = -\omega^2 A = (0.5\pi)^2 \times 8$

$= 2\pi^2 \; ms^{-2}$

(vi) Phase angle $\phi = \frac{\pi}{3} \; rads$

2. A particle is moving simple harmonic motion of period 8.0s and amplitude 5.0m. Find the:
   (i) Speed of the particle when it is 3.0m from the centre of its motion,
   (ii) Maximum speed of the particle
   (iii) Maximum acceleration

Solution

Since in the equations for speed and acceleration we have ω, it should therefore be obvious that we shall need ω in our calculations. Our first task should therefore be finding ω.

From $T = \frac{2\pi}{\omega}, \omega = \frac{2\pi}{T} = \frac{2\pi}{8.0} = 0.785$ rad s^{-1}

(i) $v = \pm\omega\sqrt{A^2 - x^2}$, *but* $A = 5.0$ *and* $x = 3.0$

$\Rightarrow v = \pm 0.785\sqrt{5^2 - 3^2}$

$= \pm 3.14 ms^{-1}$

(ii)

$V_{max} = A\omega = 0.785 \times 5 = 3.93 ms^{-1}$

(iii)

$a_{max} = \omega^2 A = 5 \times 0.785^2 = 3.08 ms^{-2}$

3. The displacement y of a mass vibrating in simple harmonic motion is given by $y = 20 \sin 10\pi t$, *where y is mm and t is in seconds. Calculate the:*
   (i) Velocity when $t = 0$,
   (ii) Acceleration when $t = 3s$

Solution

(i)

$$v = \frac{dy}{dt} = \frac{d}{dt}(20 \sin 10\pi t)$$

$$= 200\pi \cos 10\pi t$$

When $t = 0$, $v = 200\pi \cos(10\pi \times 0) = 200\pi \cos 0 = 200\pi \; mms^{-1}$

(ii) $a = \frac{dv}{dt} = \frac{d}{dt}(200\pi \cos 10\pi t) = 200\pi \frac{d}{dt}(\cos 10\pi t)$

$$= 200\pi\big(10\pi(-\sin 10\pi t)\big) = -200\pi^2 \sin 10\pi t$$

For $t = 3s$, $a = = -200\pi^2 \sin 30\pi = 0 \; mm\, s^{-2}$

Trial questions

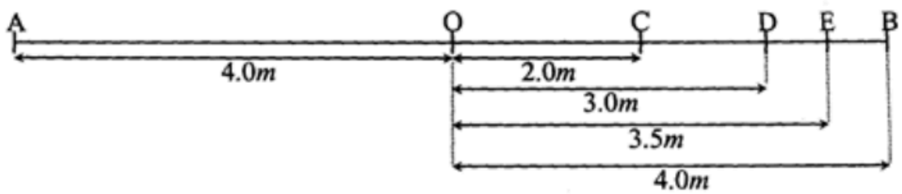
1. A particle moves with simple harmonic motion about a mean position O. When passing through two points which are 2m and 2.4m from O, the particle has speeds $3ms^{-1}$ and $1.4ms^{-1}$ respectively. Find the:
   (i) Amplitude of the motion,
   (ii) Greatest speed attained by the particle
   (iii) Maximum acceleration
   (iv) Velocity of the particle 1m from O

[Ans: $r = 2.5m$, $v_{max} = 2ms^{-1}$, $a_{max} = 5$, $v_{at\ 1m} = 4.58ms^{-1}$]

2. A particle is moving with simple harmonic motion of period 16s and amplitude 10m. Find the speed of the particle when it is 6.0m from its equilibrium position, also find how far the particle is from the equilibrium position 1.5s after passing through it, and its speed at that instant.

[Ans: $v = 3.1ms^{-1}$, $x = 5.6m$, $v' = 3.3ms^{-1}$]

3.



The figure above shows the positions of a particle as it executes simple harmonic motion of period 24s about point O, between points A and B. Find the time taken for the particle to move from :
(i) O to C
(ii) D to B
(iii) C to E

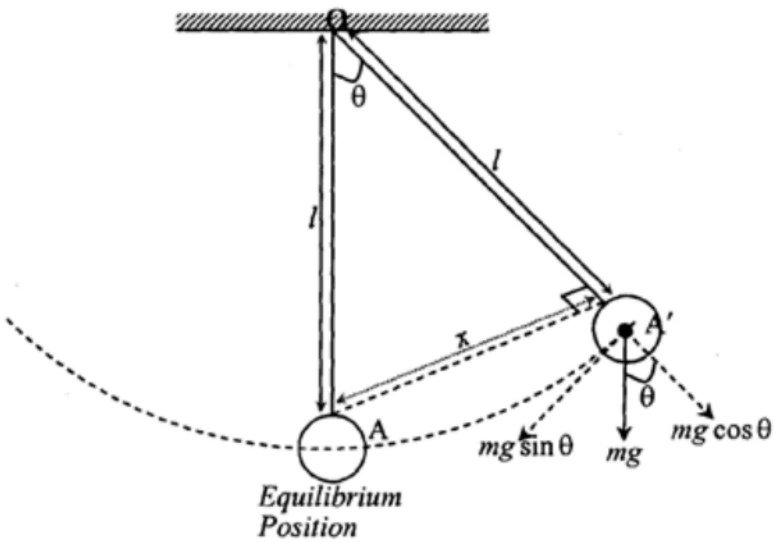
[Ans: $t_{OC} = 2.0s$, $t_{DB} = 2.8s$, $t_{CE} = 2.1s$]

4. The displacement *f* of a body moving with simple harmonic motion is given by $x = 3.5\sin(4t + 0.2)$. Find the amplitude, ω, period, T, velocity, *v at a time t=1.5s.*

Examples of oscillatory motion which approximates simple harmonic motion include;
Simple pendulum, mass on a helical spring, liquid in U-tube, piston in a gas filled cylinder, floating cylinder in a liquid.

THE SIMPLE PENDULUM

Consider the motion of a pendulum of mass m suspended at the end of a light inextensible string of length, *l,* which is fixed at the other end O. Assume that the bob is given a small displacement *x,* and that at that point, the string makes at an angle θ with its equilibrium position as shown below.



When the bob is released, a force mgsin θ acts on it and causes it to return or move towards the equilibrium position. This force is called a restoring force.

$\therefore F = -mg\sin\theta$

but for small values of $\theta, triangle\ OA'A \approx a$ right angled triangle, where $\angle OA'A \approx 90^0$

$\therefore \sin\theta = \frac{x}{l}$

$\Rightarrow F = -mg\frac{x}{l}$

But $F = ma$, $\Rightarrow ma = -mg\frac{x}{l}$ ……… (i)

The negative sign means that the acceleration acts towards the equilibrium position, the bob is moving away from the equilibrium position, and so the displacement *x is* measured in a direction opposite to the force.

From equation (i), driving through by m gives: $a = -g\frac{x}{l}$

This equation is in the form $a = -\omega^2 x, where\ \omega^2 = \frac{g}{l}$

Therefore, since $a \propto x, the$ motion of the simple pendulum bob is simple harmonic motion

$\Rightarrow \omega = \sqrt{\frac{g}{l}}$ $\quad and\ from\ T = \frac{2\pi}{\omega}$, $T = 2\pi\sqrt{\frac{l}{g}}$……..(ii)

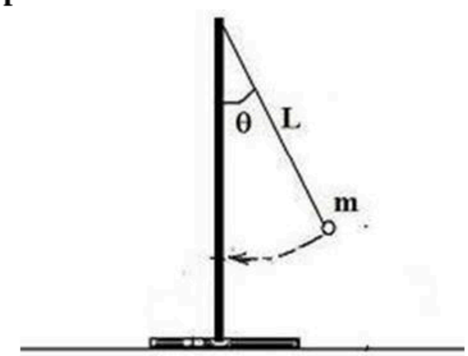
Also from $f = \frac{1}{T}$, $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

It can be seen from equation (ii) that the period of oscillation of a simple pendulum bob is independent of its mass.
It has been assumed that the displacement the is small (i.e small angle θ), and that effects due to air resistance are negligible.

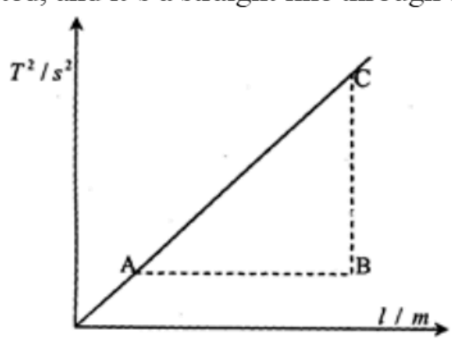
Note:
The bob eventually comes to rest due to the presence of dissipative forces arising from the air molecules. The mechanical energy of the bob is gradually changed to internal energy of the surrounding air molecules and the amplitude reduces with time.

Determination of the acceleration due to gravity g

- **Using a pendulum bob**



For different values of the length *l* of the string, time *t* for say 20 oscillations of the swinging pendulum bob is measured.
The values are tabulated including values of the period T and T^2. A graph of T^2 against *l* is plotted, and it's a straight line through the origin as shown.



$$slope,\ m = \frac{\overline{BC}}{AB} \quad \text{and}$$

$$\text{From } T = 2\pi\sqrt{\frac{l}{g}}, \qquad T^2 = \frac{4\pi^2}{g} l$$

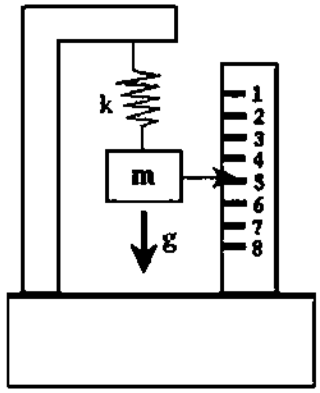
Which is in the form of $y = mx$, *where*:

$$slope,\ m = \frac{4\pi^2}{g}, \quad \Rightarrow \quad g = \frac{4\pi^2}{m}$$

However in practice, the graph in the preceding case may not pass through the origin as stated. This could be due to:

- Air resistance
- Uncertainty in the measurement of the length *l*,
- The angle of swing being big (greater than about 10^0)

- **Using a spiral spring of known force constant**



Suspend the spring from a retort stand. Attach a pointer to the spring so that it is horizontal and note its position.
A mass M is suspended from the end of the spring and the extension e it produces is noted.
The mass is then slightly pulled down and the periodic time T for the oscillations is noted.
The procedure is repeated for other masses in steps and the results recorded in a table including a column for T^2.
A graph of T^2 against e is plotted and its slope S is obtained

The slope $S = \frac{4\pi^2}{g} \Rightarrow g = \frac{4\pi^2}{S}$

Note: There may be possible errors due to
- Bad timing of the oscillations
- Incorrect reading of the extension e

Examples:
1. The period of oscillation of a pendulum on Earth is 3.50seconds. What is the period of oscillation of the same pendulum if it is taken to the moon where the acceleration due to gravity is $1.67ms^{-2}$

Solution

On Earth; $g_e = 9.81\ ms^{-2}$, $T_e = 3.50s$

While at the moon, $gm = 1.67ms^{-2}$, Tm = ?

From $T^2 = \frac{4\pi^2}{g} l,\ T_e^{\ 2} = \frac{4\pi^2}{g_e} l \ldots\ldots\ldots(i) \quad and\ T_m^{\ 2} = \frac{4\pi^2}{g_m} l$ ………(ii)

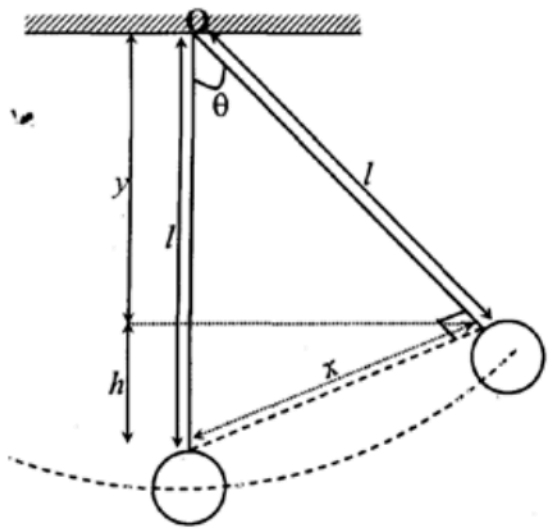
Dividing the two equations: $\frac{T_e^{\ 2}}{T_m^{\ 2}} = \frac{g_m}{g_e},\ \ T_m = \sqrt{\frac{g_e T_e^{\ 2}}{g_m}}$

$$T_m = \sqrt{\frac{9.81\times(3.51)^2}{1.67}} = 8.48s$$

2. show that the principle of conservation of mechanical energy also applies to an oscillating bob of a simple pendulum

Solution

Let the bob be displaced through a small distance, x to one side, such that the string makes a small angle θ with its initial direction



From the diagram; $\tan\theta = \frac{x}{l}$ *and* $\cos\theta = \frac{y}{l}$

But for small angles, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ *and* $\tan\theta \approx \theta$

$$\therefore \frac{y}{l} = 1 - \frac{1}{2}\theta^2 \Rightarrow y = l\left(1 - \frac{1}{2}\theta^2\right)$$

$$h = l - y = l - \left[l\left(1 - \frac{1}{2}\theta^2\right)\right] = \frac{1}{2}l\theta^2$$

but $\frac{x}{l} = \theta \Rightarrow h = \frac{1}{2}l\left(\frac{x}{l}\right)^2 = \frac{1}{2}\frac{x^2}{l}$

P.E = $mgh = mg\left[\frac{1}{2}\frac{x^2}{l}\right] = \frac{1}{2}m\frac{g}{l}x^2$

M.E = P.E + K.E $= \frac{1}{2}m\frac{g}{l}x^2 + \frac{1}{2}mv^2$

From $x = A\sin\omega t$, $v = \frac{dx}{dt} = \omega A\cos\omega t$

$$\therefore M.E = \frac{1}{2}m\frac{g}{l}(A\sin\omega t)^2 + \frac{1}{2}m(\omega A\cos\omega t)^2 = \frac{1}{2}m\frac{g}{l}sin^2\omega t + \frac{1}{2}m\omega^2A^2cos^2\omega t$$

$$= \frac{1}{2}m\frac{g}{l}A^2(sin^2\omega t + cos^2\omega t) \quad but\ sin^2\omega t + cos^2\omega t = 1$$

$$\therefore M.E = \frac{1}{2}m\frac{g}{l}A^2 = constant$$

Therefore the principle applies to an oscillating simple pendulum bob.

Trial questions

1. The table below shows data for different lengths, l and the corresponding periods of oscillation T of a simple pendulum

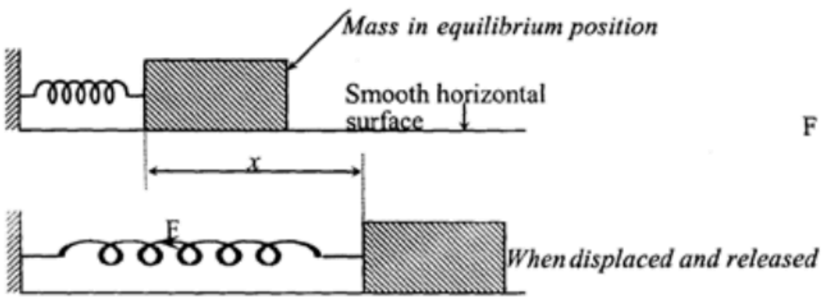
Length, l	0.20	0.300	0.40	0.50
Period, t	0.90	1.09	1.25	1.40

(i) Plot a suitable graph and use it to determine the value of the acceleration due to gravity,

(ii) Explain why the graph may not pass through the origin

2. If a pendulum has a period of 1.5s at the earth's surface. What would be its period at a height of 6000km above the Earth's surface, assuming that the radius of the earth is 6400km. [T = 2.91s]

- **Horizontal motion**



F *is the restoring force*

Consider a spring of force constant k fixed at one end, and attached to a body of mass m placed on a smooth horizontal surface on the other end. It can be shown that when the spring is slightly displaced through a distance x and then released, it executes simple harmonic motion.

From Hooke's law, the restoring force , $F \propto x \Rightarrow F = -kx$

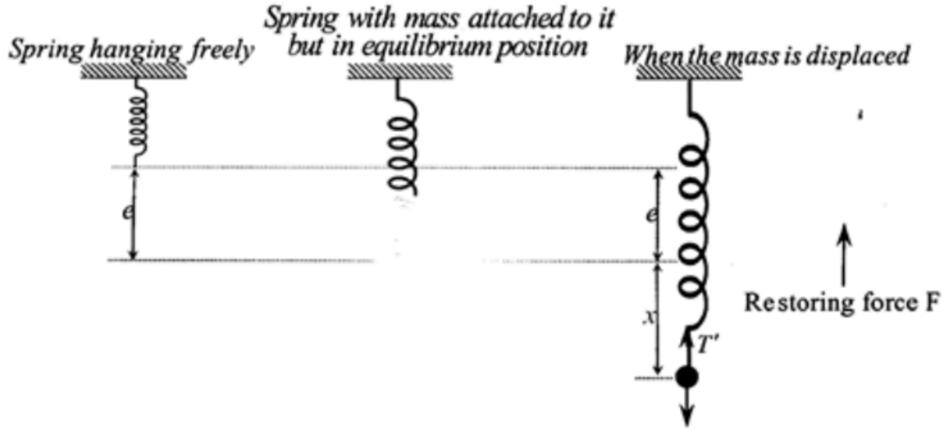
$\therefore ma = -kx \quad and\ so, \qquad a = -\frac{k}{m}x$

Therefore, since $a \propto x$, motion of the mass is simple harmonic

From $\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Frequency $= \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

- **Vertical springs**



Consider a spring of force constant k being suspended from a fixed point, and that when a body of mass m is suspended at the end of the spring, it causes an extension e on the spring. From Hooke's law, this extension is directly proportional to the tensional force in the spring.

$\therefore T \propto e \Rightarrow T = ke$

But since in the equilibrium position the body is not moving, T = mg

Thus mg = k e ……….(i)

When the body is slightly pulled downwards through a small distance x below the equilibrium position, total extension = (e + x)

Therefore, T' $= k(e + x)$

There are two forces acting on the body at this point but the resultant upward force F is the restoring force and is given by;

$F = -(T' - mg)$

$\Rightarrow F = -[k(e + x) - mg]$ but from eqn (i) mg = ke

$\therefore F = -[ke + kx - ke] = -kx \quad but\ F = ma$

$\Rightarrow ma = -kx \qquad \therefore\ a = -\frac{k}{m}x$

The equation is in the form $a = -\omega^2 x$, $\omega^2 = \frac{k}{m}$

Therefore, since $a \propto x$, motion of the mass is simple harmonic

From $\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}}$ $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

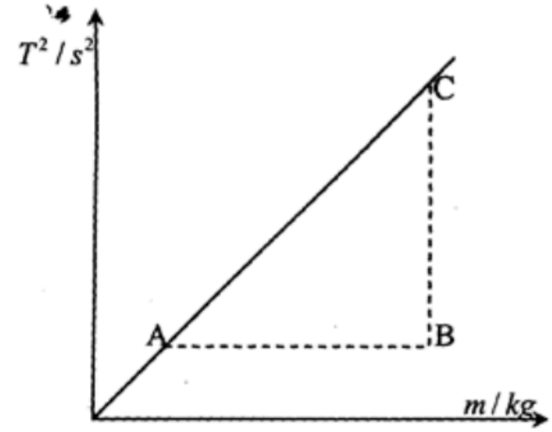
From equation (i) , $\frac{m}{k} = \frac{e}{g}$ $\qquad \Rightarrow T = 2\pi\sqrt{\frac{e}{g}}$

NB: It has been assumed that the displacement of the mass is small, effects due to air resistance are negligible, and that the mass of the spring is negligible.

Determine of the value of the spring constant

One end of the spring is clamped and a known mass m is attached to the other end of the spring. The mass is displaced, and the time *t* taken for 20 oscillations is recorded. The experiment is repeated for other known masses. The results are tabulated including values of T and T^2 .

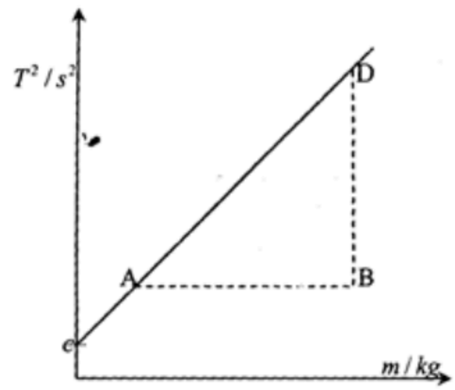
A graph of T^2 against m is plotted and it's a straight line through the origin



However the fact the spring passes through the origin is only true when the mass of the spring in question is negligible. This is however not always true in practice, and so the graph does not pass through the origin. If the mass of the spring is m_s, then the effective mass is $(m_s + m)$

$$\Rightarrow T = 2\pi\sqrt{\frac{m_s+m}{k}} \quad \text{and so } T^2 = 4\pi^2\left(\frac{m_s+m}{k}\right) = \frac{4\pi^2}{k}m + \frac{4\pi^2}{k}m_s$$

The equation is in the form $y = mx + c, such$ that if a graph of $T^2 against\ m$ is plotted, it's as shown below.



$$slope,\ s = \frac{\overline{BD}}{\overline{AB}} = \frac{4\pi^2}{k}$$

$$\Rightarrow k = \frac{4\pi^2}{s}$$

$$Also,\ Intercept,\ c = \frac{4\pi^2}{k}m_s$$

Substituting for k gives: $c = \dfrac{4\pi^2}{\left(\frac{4\pi^2}{s}\right)}m_s$

Therefore, $m_s = \dfrac{c}{s}$

The reader should therefore be able to describe an experiment to determine the mass of the spring.

Example

A helical spring gives a displacement of 5cm for a load of 500g.

(i) Calculate the period of small vertical oscillations

(ii) Find the maximum displacement produced when a mass of 80g is dropped from a height of 10cm onto a light pan attached on to the spring.

Solution

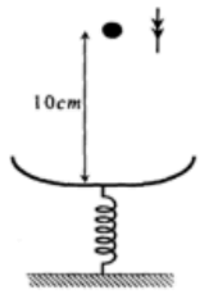
(i) $e = 5cm = 0.05m,\ \ m = 0.5kg$

But $k\,e = mg \Rightarrow k = \frac{mg}{e} = \frac{0.5\times9.81}{0.05}$

$\therefore k = 98.1\text{Nm}^{-1}$

Also, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi \times \sqrt{\frac{0.5}{98.1}} = 0.45s$

(ii)



Gravitational force lost by the body, potential energy = energy stored in the spring

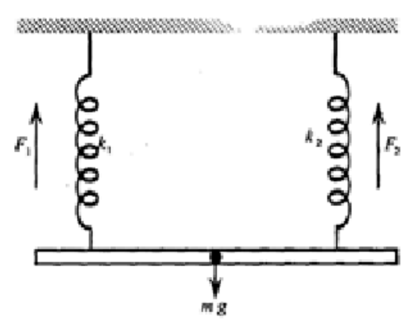
$\Rightarrow mgh = \frac{1}{2}kx^2$

$\therefore x = \sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2\times0.08\times9.81\times0.1}{98.1}} = 0.04\ m$

COMBINED SPRINGS

Case 1: Vertical springs

- Parallel combination



Consider two springs of force constants k_1 and k_2 are arranged parallel to each other, and a mass m placed at their other ends. If the mass is pulled downwards through a small displacement *x*. The extension that will be caused in each spring is the same, and will be *x*.

The restoring forces is each spring are:

$F_1 = -k_1x$ and $F_2 = k_2x$

Total restoring force, $F = F_1 + F_2 = -k_1x$ + -k_2x

Therefore , $ma = -(k_1 + k_2)x$

$\Rightarrow a = -\frac{(k_1+k_2)}{m}x$

This equation is in the form $a = -\omega^2x, where\ \ \omega^2 = \frac{k_1+k_2}{m}$

Therefore, since $a \propto x$, motion of the mass is simple harmonic.

From $\omega^2 = \frac{k_1+k_2}{m}$, $\omega = \sqrt{\frac{k_1+k_2}{m}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$

Frequency $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}}$

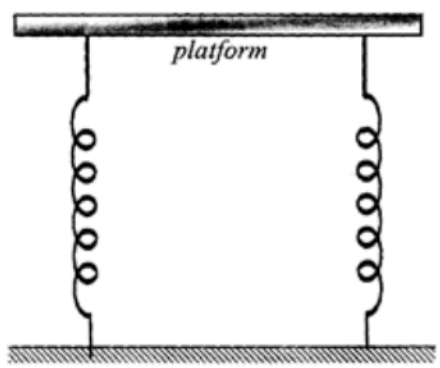
Note that if the springs are identical, then $k_1 = k_2 = k$

$$\Rightarrow a = -\frac{2k}{m}x \text{, and so } \omega^2 = \frac{2k}{m}$$

Period, $T = 2\pi\sqrt{\frac{m}{2k}}$ and frequency, $f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

Trial question

A light platform is supported by two identical springs each of force constant 40 Nm^{-1}, as shown in the diagram below.

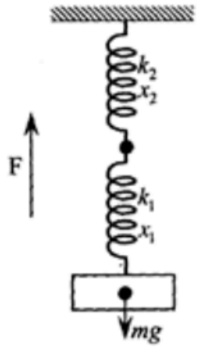


(a) What weight should be placed on the centre of the platform in order to produce a displacement of 3.0cm?

(b) With the weight in (a) above on the platform, the platform is depressed a further 1.0 cm and then released, such that it oscillates in simple harmonic motion.
   (i) Calculate the frequency of the platform
   (ii) Calculate the maximum acceleration of the platform

[Ans: (i) 2.4 N, $f = 2.9$ Hz, $a = 3.3ms^{-2}$]

- Series combination



Consider two springs of force constants k_1 and k_2 respectively connected in series and fixed at the upper end, and a known mass m fixed at the lower end. If the mass is pulled downwards through a small distance x, there will be different displacements x_1 and x_2 in the respective springs, as shown in the figure

But $x_1 + x_2 = x \ldots\ldots\ldots (i)$

However, since the springs are in series, the restoring force F in the springs is the same.

From Hooke's law,

$$F = -k_1x_1 \;\; and \; F = -k_2x_2 \;\; \Rightarrow x_1 = -\frac{F}{k_1} \;\; and \; x_2 = -\frac{F}{k_2}$$

Substituting for x_1 and x_2 $in\ equation\ (i)\ gives;$ $-\frac{F}{k_1} + -\frac{F}{k_2} = x$

$$\Rightarrow -\left(\frac{k_1+k_2}{k_1k_2}\right)F = x$$

$$\therefore F = -\left(\frac{k_1k_2}{k_1+k_2}\right)x \;\Rightarrow ma = -\left(\frac{k_1k_2}{k_1+k_2}\right)x$$

Therefore , $a = -\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)x$

This equation is in the form $a = -\omega^2 x, where\ \ \omega^2 = \frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)$

Therefore, since $a \propto x$, motion of the mass is simple harmonic.

From , $\omega^2 = \frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)$ $\qquad \omega = \sqrt{\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)}$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{m}\left(\frac{k_1+k_2}{k_1k_2}\right)}$$

Frequency $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)}$

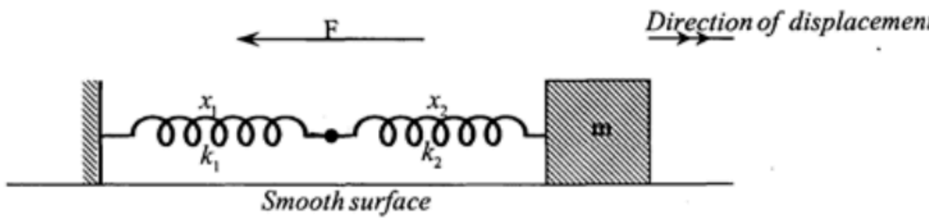
Note that if the springs are identical, $k_1 = k_2 = k$

$$\Rightarrow a = -\frac{1}{m}\left(\frac{k^2}{2k}\right)x = -\frac{1}{k}\left(\frac{k}{2m}\right)x\ , and\ \ so\ \omega^2 = \frac{k}{2m}$$

Period, $T = 2\pi\sqrt{\frac{2mk}{k^2}} = 2\pi\sqrt{\frac{2m}{k}}$ and frequency, $f = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$

Case 2: Horizontal combined springs

- Springs connected together, with the mass, m at the end



Consider two springs of force constants k_1 and k_2 connected together, fixed at one end, and a mass m attached to the other end as shown above.
Let the mass be displaced to the right as shown through a distance *x,* such that the extensions caused in the respective springs are x_1 *and* x_2.

$$x_1 + x_2 = x \dots\dots\dots (i)$$

The restoring force is the same for both springs, and if the force is F,

$$F = -k_1x_1 \quad and\ F = -k_2x_2 \quad \Rightarrow x_1 = -\frac{F}{k_1} \quad and\ x_2 = -\frac{F}{k_2}$$

Substituting for x_1 and x_2 $in\ equation\ (i)\ gives;$ $-\frac{F}{k_1} + -\frac{F}{k_2} = x$

$$\Rightarrow -\left(\frac{k_1+k_2}{k_1k_2}\right)F = x \quad , \quad ma = -\left(\frac{k_1k_2}{k_1+k_2}\right)x$$

Therefore , $a = -\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)x$

This equation is in the form $a = -\omega^2 x, where\ \ \omega^2 = \frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)$

Therefore, since $a \propto x$, motion of the mass is simple harmonic.

From , $\omega^2 = \frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)$ $\qquad \omega = \sqrt{\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)}$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{m}\left(\frac{k_1+k_2}{k_1k_2}\right)}$$

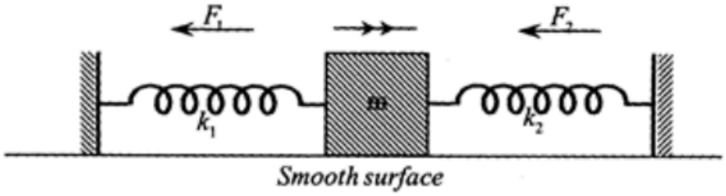
Frequency $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{m}\left(\frac{k_1k_2}{k_1+k_2}\right)}$

Note that if the springs are identical, $k_1 = k_2 = k$

$$\Rightarrow a = -\frac{1}{m}\left(\frac{k^2}{2k}\right)x = -\frac{1}{k}\left(\frac{k}{2m}\right)x\ , and\ \ so\ \omega^2 = \frac{k}{2m}$$

Period, $T = 2\pi\sqrt{\frac{2mk}{k^2}} = 2\pi\sqrt{\frac{2m}{k}}$ and frequency, $f = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$

- Springs fixed and mass placed between the springs



If the mass in the figure is displaced through a small displaced through a small distance *x,* say to the right as shown above, the spring of force constant k_1 is extended by x while the spring of force constant k_2 is compressed by the same distance x. The restoring forces in either springs are different, are different, but are all in the same direction, opposite to the direction in which the mass has been displaced.

$$F = -k_1x_1 \;\; and \; F = -k_2x_2$$

The total restoring force F on the mass is the sum of these two forces.

$$\Rightarrow F = F_1 + F_2 = -k_1x_1 + -k_2x_2$$

$$\therefore ma = -(k_1 + k_2)x \quad \Rightarrow a = -\left(\frac{k_1+k_2}{m}\right)x$$

This equation is in the form $a = -\omega^2 x, where \;\; \omega^2 = \frac{k_1+k_2}{m}$

Therefore, since $a \propto x$, motion of the mass is simple harmonic

From $\omega^2 = \frac{k_1+k_2}{m}$, $\omega = \sqrt{\frac{k_1+k_2}{m}}$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$

Frequency, $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}}$

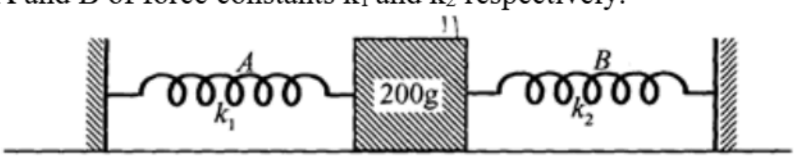
It should be noted that if the springs are identical, then; $k_1 = k_2 = k$

$$\Rightarrow a = -\frac{2k}{m}x \;, and \; so \; \omega^2 = \frac{2k}{m}$$

Period, $T = 2\pi\sqrt{\frac{m}{2k}}$ $\quad and \;\; Frequency, f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

Example

The figure below shows a mass of 200g resting on a smooth horizontal table, attached to tow springs A and B of force constants k_1 and k_2 respectively.



The block is pulled through a distance of 8cm to the right and then released.

(i) Show that the mass oscillates with simple harmonic motion, and find the frequency of oscillation, if $k_1 = 120\ Nm^{-1}$ and $k_2 = 200Nm^{-1}$

(ii) Find the new amplitude of oscillation when a mass of 120g is dropped vertically onto the block as the block passes the equilibrium position. Assume that the mass sticks to the block.

Solution

(i)

$$\omega = \sqrt{\frac{k_1+k_2}{m}} = \sqrt{\frac{120+200}{0.2}} = 40\ rads^{-1}$$

$$f = \frac{\omega}{2\pi} = \frac{40}{2\times3.14} = 6.37 \text{ Hz}$$

(ii) Let A = 8cm = original amplitude, $A' = New\ amplitude, m = 200\text{g}\ , m' = 120\text{g}$

Also let v_{max} = maxmum velocity at equilibrium position for m,

And $v'_{max} = maximum\ velocity\ at\ equillibrium\ position\ for\ m + m'$

$$v'_{max} = \omega A = 40 \times (8 \times 10^{-2}) = 3.2\text{ms}^{-1}$$

From the conservation of momentum, $mv_{\ max} = (m + m')v'_{max}$

$$\therefore 0.2 \times 3.2 = 0.32v'_{max} \Rightarrow v'_{max} = 2\text{ms}^{-1}$$

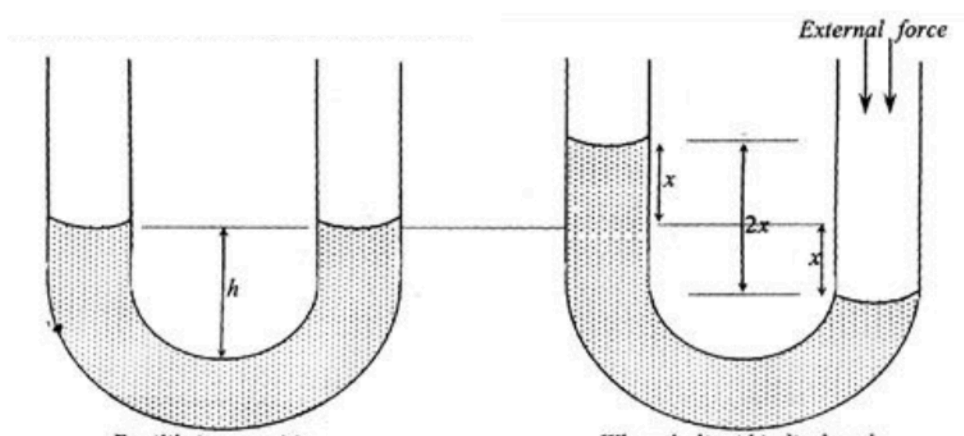
From conservation of energy, New K.E $_{max}$ = Elastic P.E $_{max}$

$$\therefore \frac{1}{2}mv'^{\,2}_{max} = \frac{1}{2}(k_1 + k_2)x^2 = \frac{1}{2}(k_1 + k_2)(A')^2$$

$$\frac{1}{2} \times 3.2 \times 2^2 = \frac{1}{2} \times 320 \times (A')^2$$

$$\therefore A' = 6.32\ cm$$

OSCILLATION OF A LIQUID IN A U-TUBE



Equilibrium position *When the liquid is displaced*

Consider a liquid of density ρ in a U-tube of uniform cross sectional area A, that the liquid is at a height h in either limb of the tube.

Total length l of the tube occupied by the liquids is $2h$

Volume of the liquid, V = Al =$2hA$

Mass of liquid, m = Vρ

$\Rightarrow m = 2hA\rho$ ………….. (i)

If an external force F is applied to one side of the tube for example by blowing gently, the levels of the liquid will for a short time oscillate above the initial equilibrium position.

There is a resulting excess height of $2x$ of liquid in one limb as compared to the other due to the external force. There is therefore a resulting excess pressure P in the limb.

$$P = 2x \times \rho \times \text{g} = 2x\rho\text{g}$$

But Force = pressure × area $\Rightarrow F = -2x\rho\text{g}$

$\therefore ma = -2x\rho g \Rightarrow a = -\frac{2x\rho g A}{m} = -\frac{g}{h}x$

Substituting for m from equation (i) above gives: $a = -\frac{2x\rho g A}{2hA\rho} = -\frac{g}{h}x$

$\Rightarrow a = -\frac{g}{h}x$

This equation is in the form $a = -\omega^2 x, where \ \omega^2 = \frac{g}{h}$

Therefore, since $a \propto x$, motion of the mass is simple harmonic

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$

And $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$

Note: The oscillations of the liquid are heavily slowed down due to viscosity, and so die out in a short time.

It is assumed that the displacement of the liquid is small, and the viscous forces or viscosity are negligible.

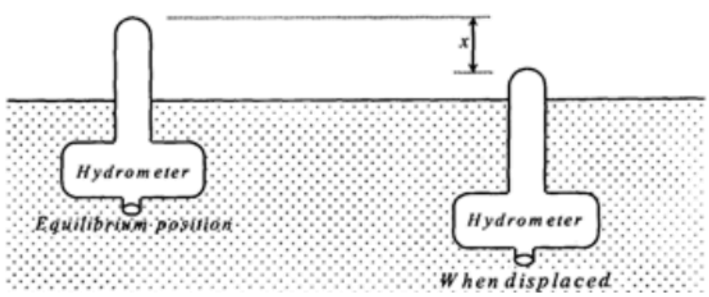
Example

A hydrometer is made to float in a liquid and there after given a small vertical displacement after which it is released. Show that the hydrometer executes simple harmonic motion.

Solution

Let the hydrometer be displaced through a small displacement *x* downwards from the equilibrium position. Also, let the cross sectional area of the stem for the hydrometer be A.

Assuming that in the equilibrium position, volume of the hydrometer in the liquid is V, then; from the law of floatation, a floating body displaces its own weight of the liquid in which it floats.



In equilibrium position, weight of the liquid displaced by the hydrometer = mg = $V\rho g$, ρ is the density of the liquid.

When the hydrometer is slightly displaced down wards, there is an additional volume A*x onto* the hydrometer.

$\Rightarrow$ New volume of the hydrometer $= V + Ax$

Therefore weight of liquid displaced = $(V + Ax)\rho g$

This implies that up thrust U = weight of liquid displaced = $(V + Ax)\rho g$

Restoring force F = $-(U - mg) = -((V + Ax)\rho g - mg) = -(V\rho g + Ax\rho g - mg)$

But mg = $V\rho g \quad \Rightarrow F = -(V\rho g + Ax\rho g - V\rho g)$

$\therefore F = -Ax\rho g$

Therefore, $ma = -Ax\rho g$

Substituting for m gives; $(V\rho g)a = -Ax\rho g$

$\therefore a = -\frac{A\rho}{V}x$

This equation is in the form $a = -\omega^2 x, where\ \omega^2 = \frac{A\rho}{V}$

Therefore, since $a \propto x$, motion of the mass is simple harmonic

$\Rightarrow \omega = \sqrt{\frac{A\rho}{V}}$ thus $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{V}{A\rho}}$

Frequency, $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{A\rho}{V}}$

Trial questions

1. A volume v of air, pressure P is contained in a cylindrical vessel of cross sectional area A by a frictionless air tight piston of mass m. The piston is slightly forced downwards, and then released. Show that the piston oscillates with simple harmonic motion of period, T given by:

$$T = \frac{2\pi}{A}\sqrt{\frac{mv}{P}}$$

2. A uniform cylindrical rod floats in a fluid with a height x immersed. Show that when the rod is given a small vertical displacement into the liquid and then released, it executes simple harmonic motion of period , $T = 2\pi\sqrt{\frac{x}{g}}$ where g is the acceleration due to gravity.
3. A platform moves up and down with simple harmonic motion of period T and amplitude A. A small particle of mass m is placed on the platform.
   (i) Find an expression in terms of g, T and x for the reaction of the platform on the particle when the particle is at a displacement x from its mean position.
   (ii) If the platform vibrates in simple harmonic motion with a period of 0.5s, calculate the maximum amplitude for the motion, which allows the particle to remain in contact with the platform throughout the motion.

ENERGY OF A BODY IN SIMPLE HARMONIC MOTION

When a body is oscillating with simple harmonic motion, there is an interchange of energy between kinetic and potential forms. However, for as long as no work is done against dissipative forces, total energy (mechanical energy) of a body remains constant.

From $v^2 = \omega^2(A^2 - x^2)$;

Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2)$

Potential energy $= \int F\,dx = \int kx\,dx = \frac{1}{2}kx^2$

But , $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2$

Thus potential energy $= \frac{1}{2}m\omega^2 x^2$

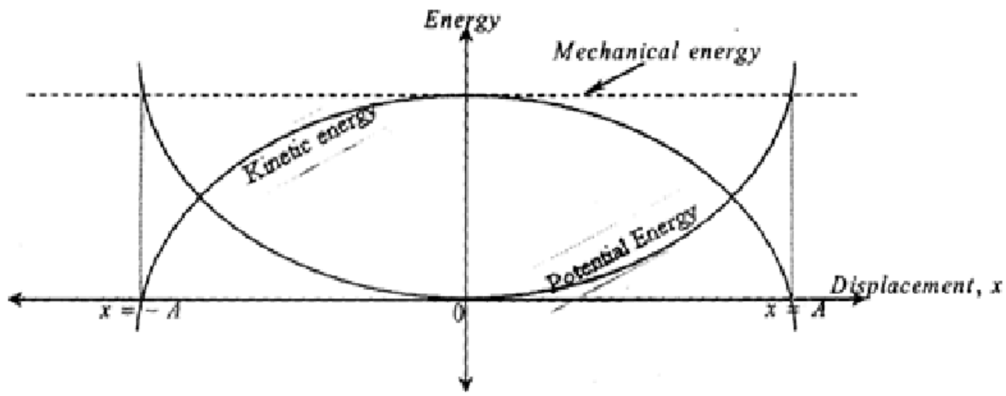
Mechanical energy = Kinetic energy + Potential energy

$= \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$

$$= \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2x^2$$
$$= \frac{1}{2}m\omega^2A^2$$

It can be seen from the equations of kinetic energy, potential energy, and mechanical energy that mechanical energy is a constant, but at the center of oscillations i.e when $x = 0$, potential energy is zero, and the velocity is maximum, implying that kinetic energy is maximum such that all the body's energy appears as its kinetic energy.
At the position of maximum displacement, x *is* maximum, and so potential energy is maximum, but since velocity is zero, kinetic energy is also zero, such that all the body's energy appears as potential energy.

Graph showing variation of energy with displacement for a body moving in s.h.m

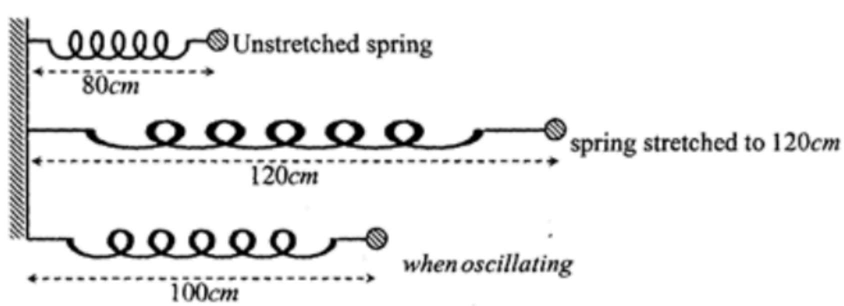


Example
A ball of mass 20g is attached to an elastic spring of natural length 80cm with one end fixed to a ridge support. The spring is stretched to 120 cm along a horizontal frictionless table and then released. The spring is of force constant 400 Nm^{-1}. For an instant when the ball is 100cm from the fixed point, calculate the;

(i) Kinetic energy of the ball
(ii) Potential energy of the ball
(iii) Velocity of the ball

Solution

k = 400 Nm^{-1}, r = 120 – 80 = 40cm = 0.4m , x = 100 – 80 = 20 cm = 0.2m



(i) Kinetic energy = $\frac{1}{2}m\omega^2(r^2 - x^2)$ but $\omega^2 = \frac{k}{m}$

$\Rightarrow K.e = \frac{1}{2}k(r^2 - x^2) = \frac{1}{2} \times 400 \times (0.4^2 - 0.2^2) = 24$ Joules

(ii) Potential energy = $\frac{1}{2}m\omega^2 x^2$ but $\omega^2 = \frac{k}{m}$

$\Rightarrow P.e = \frac{1}{2}kx^2 = \frac{1}{2} \times 400 \times 0.2^2 = 8$ Joules

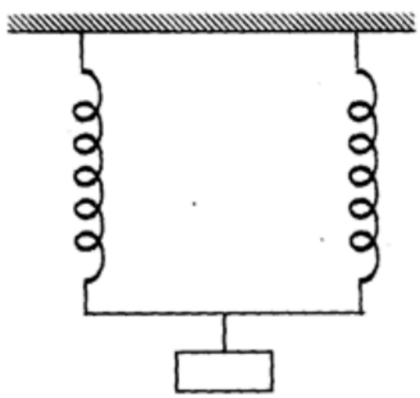
(iii) Kinetic energy = $\frac{1}{2}mv^2$

$\Rightarrow 24 = \frac{1}{2} \times \frac{20}{100} \times v^2$

$\therefore v = 15.49ms^{-1}$

Trial questions

1. A light spring is suspended from a rigid support and its free end carries a mass of 0.40 kg which produces an extension of 0.060m on the spring. The mass is slightly pulled down an additional 0.060m and released, such that the mass oscillates with simple harmonic motion. Calculate the kinetic energy of the mass as it passes through the midpoint of its motion. [Ans : 0.12J]
2. Two springs each of spring constants $200Nm^{-1}$ and $100Nm^{-1}$ respectively are suspended side by side on a rigid support. A mass of 2.0kg is suspended as shown below



Calculate the:

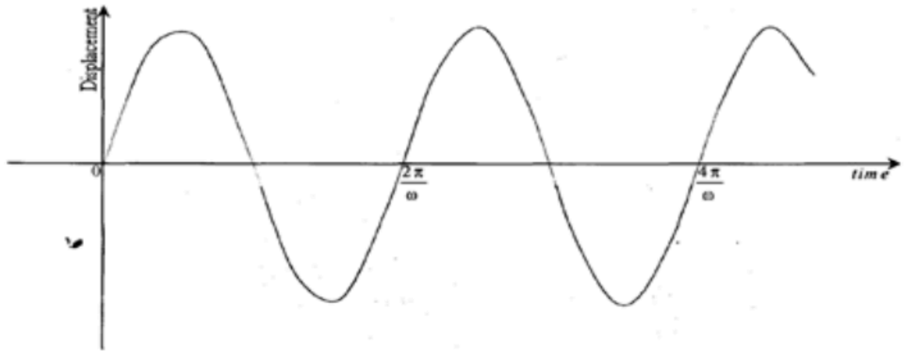
(i) Extension produced
(ii) Tension in each spring
(iii) Energy stored in the springs
(iv) Frequency of the small oscillations that would result when the mass is given a small vertical displacement. [Ans: $e = 6.52 \times 10^{-2}m$, $T_1 = 13.08N$, $T_2 = 6.54N$, Energy = 1.283 Joules, f =1.95Hz]

TYPES OF OSCILLATIONS

- Free oscillations

These are oscillations which occur in absence of any dissipative forces such as air resistance, friction, e.t.c.

The system oscillates indefinitely and its energy remains constant, implying that no energy is lost to the surrounding. The amplitude of the oscillations also remains constant.



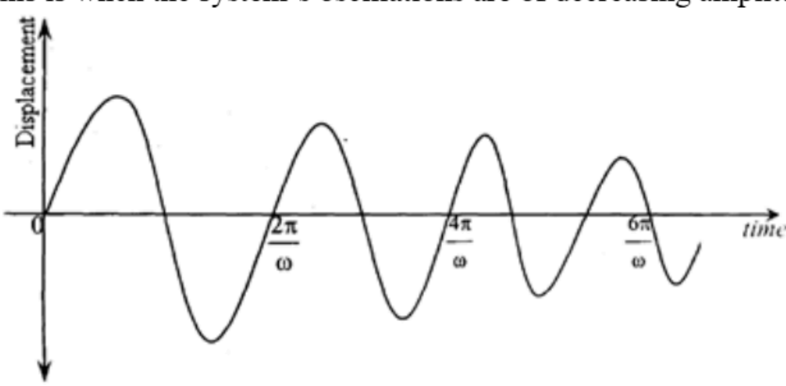
- Damped oscillations

These are oscillations in which the oscillating system loses energy to the surrounding due to dissipative forces such as friction, air resistance, e.t.c. The amplitude of the oscillation decreases with time.

Damped oscillations can further be categorized into three situations:

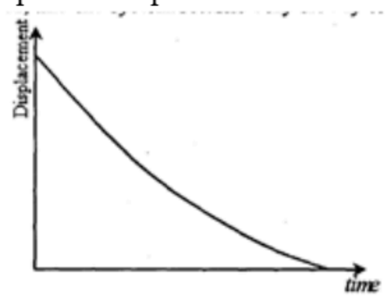
(i) Slightly damped (under damped motion)

This is when the system's oscillations are of decreasing amplitude



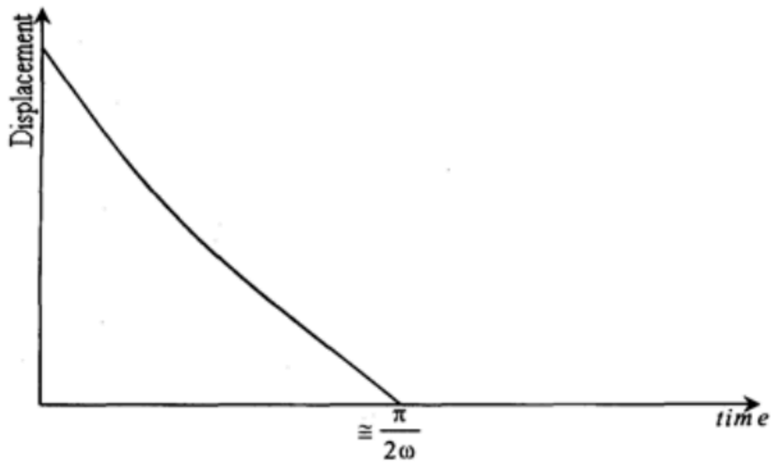
(ii) Heavily damped (over damped motion)

For this case, no oscillations occur, and the system returns very slowly to its equilibrium position.



(iii) Critically damped

For critically damped motion, the time taken for the displacement to become zero is minimum i.e the time is approximately $\frac{1}{4}T = \frac{\pi}{\omega}$. This type of motion is applied in pointer instruments such as moving coil galvanometer, voltmeter e.t.c. This ensures that the pointer takes as little time as possible to settle at the equilibrium position, such that a reading can be taken. For such instruments, the damping is provided by electromagnetic forces.



- Forced oscillations

These are oscillations where the system is subjected to an external periodic force, so as to set it oscillating for the required time.

Due to damping, free oscillations eventually die out. If it is required that the motion of the system remains, then an external periodic force must be supplied to the system.

When the applied periodic force has the same frequency of oscillation as the system, the system absorbs a maximum amount of energy. This is called resonance.

Trial questions

1. A light elastic string hangs vertically with its upper fixed end and a body attached to its lower end. The body rests initially in equilibrium with the string stretched 5cm beyond its unstretched length. The body is pulled down a further distance of 10cm and released from rest. Show that the period of the ensuring motion is given by: $T = \frac{1}{21}\left[2\pi + 3\sqrt{3}\right]s$
2. A particle describes simple harmonic motion about a point O as centre, and the amplitude of the motion is *a* metres. Given that the period of the motion is $\frac{\pi}{4}$ seconds and that the maximum speed of the particle is 16ms^{-1}, find ;
   (i) The speed of the particle at a point a distance of $\frac{1}{2}a$ from O
   (ii) The time taken to travel directly from O to the point in (i) above

   [Ans: v = 13,86 ms^{-1} , $t = 6.5 \times 10^{-2}s$]
3. A particle P describes simple harmonic motion, making three complete oscillations per second. At a certain instant is at the point O and is moving at its maximum speed of 5ms^{-1}.
   (i) Find the speed of P 0.05 seconds after it passes through O,

(ii) If after passing through O, P first comes to instantaneous rest at a point A, find the average speed of P as it moves from O to A.

[Ans: (i) 2.94 ms^{-1} (ii) 3.18ms^{-1}]

4. A particle is released from rest at a point A at a time $t = 0$ and performs SHM about a mean position B. The particle just returns to A during each oscillation. Given that AB = $2\sqrt{2}$m, and that the particle passes through B with a speed of $\pi\sqrt{2}$ ms^{-1}, find the value of t when the particle is first travelling with a speed of π ms^{-1}. How far from B is the particle then? [Ans: t = 0.5s, $x = 2$m]
5. The effect of waves on an empty oil drum floating in a sea is to make it bob up and down with SHM. If the drum encounters 20 waves per minute and for each wave the vertical distance from peak to trough is 80 cm, find the amplitude and period of the motion and the maximum speed of the drum. [Ans: 40cm , 3s , 0.84ms^{-1}]
6. A horizontal spring of force constant 200Nm^{-1} fixed at one end has a mass of 2kg attached to one end has a mass of 2kg attached to the free end and resting on the smooth horizontal surface. The mass is pulled through a distance of 4.0 cm and released.
Calculate the:
(i) Angular speed
(ii) Maximum velocity attained by the vibrating body
(iii) Acceleration when the body is half way towards the centre from its initial position.
[Ans: (i) 10 rad s^{-1} (ii) 0.4 ms^{-1} (iii) 2ms^{-2}]
7. A particle executing simple harmonic motion vibrates in a straight line. Given that the speeds of the particle are 4 ms^{-1} and 2 ms^{-1} when the particle is 3cm and 6cm respectively from the equilibrium position. Calculate the:
(i) Amplitude of oscillation
(ii) Frequency of the particle [Ans: (i) $6.7 \times 10^{-2} m$ (ii) 10.68 Hz]
8. A body of mass 1kg moving with simple harmonic motion has speeds of 5 ms^{-1} and 3 ms^{-1} when it is at distances of 0.10m and 0.20 m respectively from its equilibrium position. Find the amplitude of the motion [Ans: 0.24 m]
9. A mass of 1.0 kg is hung from two springs S_1 and S_2 connected in series and connected to the ceiling. The force constants of the springs are 100Nm^{-1} and 200Nm^{-1} respectively. Find the
(i) Extension produced by the combination
(ii) Frequency of oscillation of the mass if it is pulled downwards through a small distance and released [Ans: 0.098m, 0.1472m, 1.3 Hz]
10. A piston in the car engine performs simple harmonic motion of frequency 12.5Hz. If the mass of the piston is 0.5kg and its amplitude of vibration is 45mm, Find the maximum force on the piston [Ans: 138.79N]

CHAPTER 13: THERMOMETRY

THERMOMETRIC SUBSTANCES

A thermometric substance is a substance which has a property of changing with change in temperature.

Examples of thermometric substances

- Resistance of a wire.
- E.m.f of a thermocouple.
- Length of a liquid column.
- Pressure of a fixed mass of a gas.
- Volume of a fixed mass of a gas.
- Wavelength of radiant electromagnetic waves.

Properties of a thermometric substance

- It should change appreciably with temperature.
- It should change linearly with temperature.
- It should be measured over a wide temperature range.

TEMPERATURE SCALES

Temperature is the degree of hotness or coldness of a substance. Temperature scale is defined by two fixed points. That is, the upper and the lower fixed points.

For a Celsius scale, the ice point (0°C) is the lower fixed point while the steam point (100°C) is the upper fixed point. By definition, **fixed points** are the two standard degrees of hotness chosen when establishing a temperature scale.

For a thermodynamic scale (Kelvin or absolute scale), the absolute zero point (0 K or -273°C) is the lower fixed point while the triple point of water (273.16K or 0.16°C) is the upper fixed point. By definition, **triple point of water** is the unique temperature where all the three states of water (pure water, pure melting ice, and pure saturated vapor) co-exist in equilibrium.

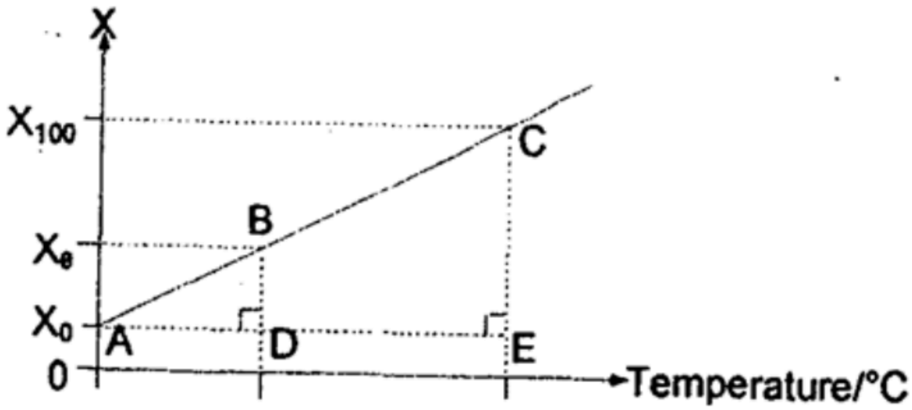
Requirements for establishing a temperature scale

- Thermometric property,
- Value of the property at fixed points,
- Fundamental interval between the fixed points.

- Fixed points.

Steps taken to establish a Celsius scale

A thermometric property, X is selected. The value of the thermometric property, X_{100} at steam point is obtained. The value of the thermometric property, X_0 at ice point is also obtained. More so, the value of the thermometric property $\boldsymbol{X_\theta}$ at the unknown temperature $\boldsymbol{\theta}$ is obtained. A graph of the thermometric property reading against temperature is plotted and is a straight line graph as shown below.



For a straight line, the unknown temperature θ can be obtained as follows

$$(Slope\ AB) = (slope\ AC)$$

$$\frac{BD}{AD} = \frac{CE}{AE}$$

$$\frac{X_\theta - X_0}{(\theta - 0)} = \frac{X_{100} - X_0}{(100 - 0)}$$

$$(100 - 0)(X_\theta - X_0) = (\theta - 0)(X_{100} - X_0)$$

$$(100)(X_\theta - X_0) = \theta(X_{100} - X_0)$$

$$\theta = \left[\frac{X_\theta - X_0}{X_{100} - X_0}\right] \times 100^o\text{C}$$

Steps taken to establish a thermodynamic scale

A thermometric property, X is selected. The value of the thermometric property, X_{tr} at triple point of water is obtained. Also, the value of the thermometric property X_T at the unknown temperature T is obtained. The unknown temperature T is then obtained from

$$T = \left(\frac{X_T}{X_{tr}}\right) \times 273.16K$$

Disagreements between scales

Scales of temperatures based on different thermometric properties may not agree because different thermometric properties depend differently on temperature. However, they may give approximately the same readings near the fixed points because thermometers are calibrated to give the same reading at fixed points.

Another reason for the disagreements between temperature scales is that the sources of error vary with the type of thermometer being used.

LIQUID IN GLASS THERMOMETERS

Thermometric property is length of the liquid column. The most common type of this thermometer is mercury in glass thermometer where the thermometric property is the length of the mercury column.

If l_θ is the length of the mercury column at θ in °C.

l_0 is the length of the mercury column at a 0°C.

l_{100} is the length of the mercury column at a 100°C

$$\theta = \left[\frac{l_\theta - l_0}{l_{100} - l_0}\right] \times 100°C$$

Why mercury is preferred in Liquid in glass thermometers

- It's opaque hence easily seen,
- It doesn't wet the glass wails,
- It's a good conductor. Therefore, it can rapidly take up the temperature of the surroundings.

Advantages of mercury in glass thermometers

- It's easy and cheap to produce,
- It's portable,
- It's opaque,
- It doesn't wet the glass.
- Temperature is directly read from the thermometer,

Disadvantages of Liquid in glass thermometers

- It's not accurate,
- Its sensitivity is low at low temperature. Therefore, it's not easy to notice temperature changes at low temperatures.

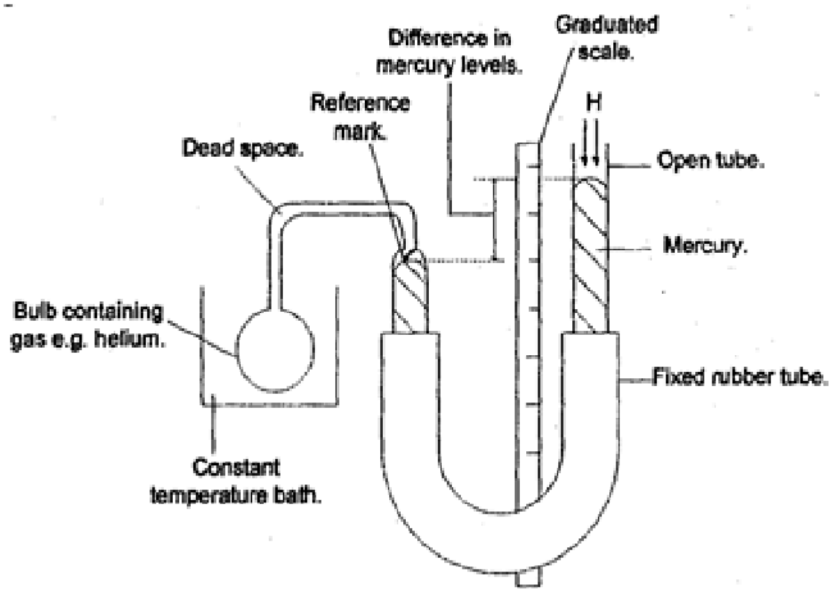
Why Liquid in glass thermometers are not used for accurate work

Though they are simple to use and cheap to buy, liquid in glass thermometers can't be used to produce accurate work. Below are some of the reasons why.

- It has a non-uniform bore which limits accuracy.
- The glass expands and contracts and can therefore take a long time to reach its correct size and therefore spoils calibration,
- Parallax errors prevent the scale from being read accurately,
- The accuracy of the calibration depends on whether the thermometer is upright or not and on how much the stem is exposed.

CONSTANT VOLUME GAS THERMOMETERS

The thermometric property used is pressure of a fixed mass of a gas.



Definition of a Celsius scale using a constant volume gas thermometer

The bulb is immersed in ice-water mixture. Time is allowed for the gas inside the bulb to come to the temperature of the bath. The open tube is then moved either upwards or downwards such that the level of the mercury in the closed tube reaches the reference mark. The difference in the levels of mercury, h_0 is measured using a graduated scale. If P_0 is the pressure at ice point, then $P_0 = (h_0 + H)$.

The bulb is then transferred to the steam bath at 1 atmosphere (standard atmospheric pressure). The procedure is repeated and the difference in the levels of mercury h_{100} noted. If P_{100} is the pressure at 100°C (steam point), then $P_{100} = (h_{100} + H)$. The bulb is then transferred to a bath of temperature, θ. The procedure is repeated and the difference in the levels of mercury h_θ noted. If P_θ is the pressure at θ°C, then $P_\theta = (h_\theta + H)$. The unknown temperature

$$\theta = \left[\frac{P_\theta - P_0}{P_{100} - P_0}\right] \times 100°\text{C}$$

Definition of a thermodynamic scale using a constant volume gas thermometer

The bulb is immersed in a bath maintained at triple point of water. Time is allowed for the gas inside the bulb to come to the temperature of the bath.

The open tube is then moved either upwards or downwards such that the level of the mercury in the closed tube reaches the reference mark. The difference in the levels of mercury, h_0 is measured using a graduated scale. If P_{tr} is the pressure at triple point of water, then $P_{tr} = (h_{tr} + H)$.

The bulb is then transferred to a bath at a temperature, T. The procedure is repeated and the difference in the levels of mercury, h_T noted. If P_T is the pressure at a temperature T, then $P_T = (h_T + H)$. The unknown temperature

$$T = \left(\frac{P_T}{P_{tr}}\right) \times 273.16K$$

Advantages of constant volume gas thermometer

- It can be used over a wide range of temperatures (270°to 1500°C).
- It's very accurate. Therefore, it can be used as a standard to calibrate other thermometers.
- It's very sensitive. Therefore, it can be used to measure very low temperatures.

Disadvantages of a constant volume gas thermometer

- It's bulky and cumbersome to operate. Therefore, one has to be careful while operating it.
- It's slow to respond to temperature changes. Therefore, it can't be used to measure rapidly changing temperatures.
- It can't be used to measure temperature at a point.

- It doesn't give direct readings.

Sources of errors in a constant volume gas thermometer

- The bulb expands. The thermal expansion of the bulb changes the volume of the gas which must be constant. This gives rise to errors,
- The air is not an ideal gas. The temperature of the gas in the dead space is not the same as that in the bulb. So, the temperature of the gas may not be uniform,
- Capillary effects at the mercury surfaces. The meniscus is convex upwards.

The three corrections that need to be made in a constant volume gas thermometer

- The bulb must be made of hard glass (e.g. Pyrex) to limit thermal expansion.
- The dead space should be made narrow.
- The manometer tube should be narrowed.

NOTE:

(1). The advantage of using gases in thermometers is because all gases have the same coefficient of expansion as long as the gas pressure is low.

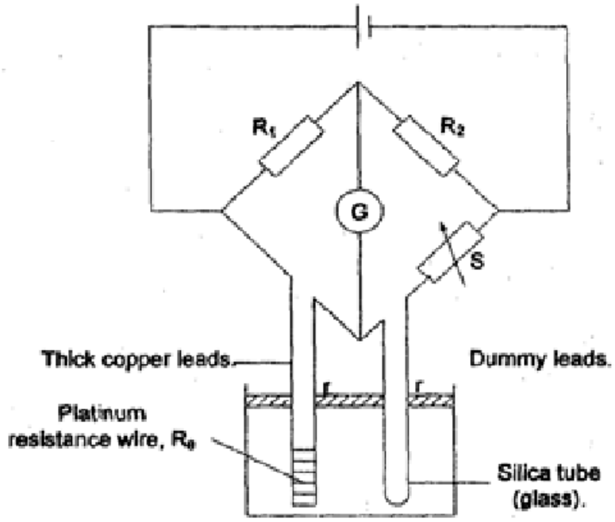
(2). Constant pressure thermometers are rarely used compared to constant volume gas thermometers. This is because in constant pressure thermometers, the changes in gas volume are usually too large that it's only convenient to use constant volume gas thermometers.

(3). Constant volume gas thermometers are used to calibrate Dl-ier thermometers for scientific work all over the world. This is because if the density of the gas is sufficiently low, all gas temperatures will yield exactly the same temperature thus making them very accurate and therefore used to calibrate other thermometers.

(4). A liquid in glass thermometer can be adjusted to a constant volume gas thermometer by suitably adjusting the degrees on the glass.

PLATINUM RESISTANCE THERMOMETERS

The thermometric property used is resistance of a platinum wire.



Definition of a Celsius scale using a platinum resistance thermometer

Resistance R_1 is made equal to R_2. The resistance of the dummy leads r is equal to the resistance of the copper leads r. The resistance R_s of S is adjusted until no current flows through the galvanometer G. At balance,

$$\frac{R_1}{R_2} = \frac{(R_\theta + r)}{(R_S + r)}$$

$$since\ R_1 = R_2,\ then\ R_\theta = R_s$$

The silica tube is immersed in ice-water mixture and the resistance R_0 is measured at ice-point. The tube is then immersed in a steam and the resistance R_{100} at steam point is measured. The tube is now brought into contact with the body of unknown temperature θ and the resistance R_θ at that temperature is measured. The unknown temperature

$$\theta = \left[\frac{R_\theta - R_0}{R_{100} - R_0}\right] \times 100°\text{C}$$

Definition of the thermodynamic scale using a platinum resistance thermometer

Resistance R_1 is made equal to R_2. The resistance of the dummy leads r is equal to the resistance of the copper leads r. The resistance R_s of S is adjusted until no current flows through the galvanometer G. At balance,

$$\frac{R_1}{R_2} = \frac{(R_T + r)}{(R_S + r)}$$

$$since\ R_1 = R_2,\ then\ R_T = R_s$$

The silica tube is immersed in bath maintained at triple point of water and the resistance R_{tr} is measured at triple point of water. The tube is now brought into contact with the body of unknown temperature T and the resistance R_T at that temperature is measured. The unknown temperature

$$T = \left(\frac{P_T}{P_{tr}}\right) \times 273.16K$$

Properties of materials used in platinum resistance thermometers

- The resistance should vary linearly with temperature.
- The material of the wire chosen should have a high temperature coefficient of resistance such that there's a measurable change in resistance.

Advantages of a platinum resistance thermometer

- It can be used to measure small steady temperature differences
- It's used over a wide temperature range (-2000°C to -1200^{0}C).
- It's very accurate but not as a constant volume gas thermometer.
- It's less cumbersome than a constant volume gas thermometer.

Disadvantages of a platinum resistance thermometer

- It can't be used to measure rapidly changing temperatures,
- It can't measure temperatures at a point due to the large size of silica tube.

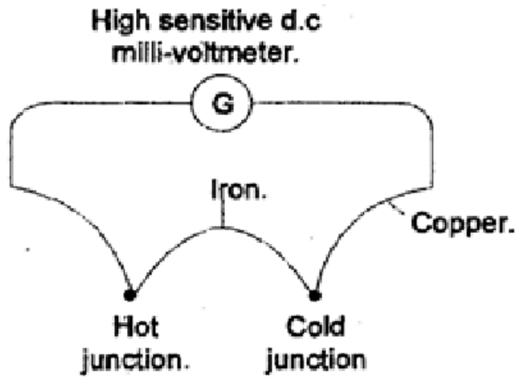
THERMOCOUPLE (THERMOELECTRIC) THERMOMETERS

The thermometric property used is e.m.f of a thermocouple.

Need for calibration of a thermocouple thermometer

When two different metals are in contact, an e.m.f is set up at the point of contact and the magnitude of this e.m.f depends on the temperature at the junction of the two metals. However, this arrangement gives a less accurate value of e.m.f. Therefore, for greater accuracy, calibration is done by adding a second junction which is kept in an ice-water mixture at 0°C and the hot junction acts as the temperature measuring junction.

Calibration of a thermocouple thermometer.

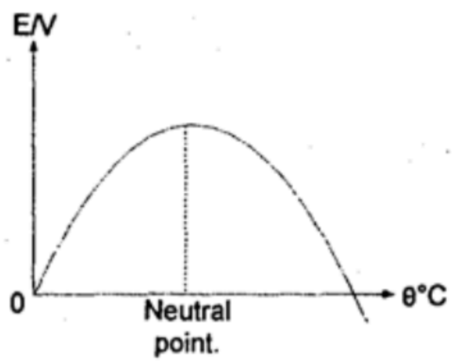


The e.m.f at the cold junction is always the same and the meter is adjusted to allow for this e.m.f. The hot junction is brought into contact with a hot substance and the reading E on the meter noted.

The temperature θ of the test substance is then obtained using a constant volume gas thermometer defined for a Celsius scale and the value of this temperature is noted.

The procedure is repeated for different hot substances and for each substance under test, the value of temperature θ in °C and e.m.f E in volts is noted.

The results are tabulated and a graph of e.m.f E against temperature θ is plotted. This graph constitutes the calibration curve and is as follows.



If the cold junction is at 0°C, the value of e.m.f E as a function of θ is given by

$$E = a\theta + b\theta^2$$

Where *a and b* are constants whose values depend on the type of the two metals used.

Definition of the Celsius scale using a thermocouple thermometer

$$\theta = \left[\frac{E_\theta - E_0}{E_{100} - E_0}\right] \times 100°C$$

Where E_θ is the e.m.f of the substance at a temperature θ in °C.

E_0 is the e.m.f of the substance at a 0°C.

E_{100} is the e.m.f of the substance at a 100°C.

Definition of the thermodynamic scale using a thermocouple thermometer

$$T = \left(\frac{E_T}{E_{tr}}\right) \times 273.16K$$

Where E_T is the e.m.f of the substance at a temperature T in kelvin.

E_{tr} is the e.m.f of the substance at triple point of water.

Advantages of thermocouple thermometers over platinum resistance thermometers

- Thermocouples have a wider temperature range [(-200°C to -1600°C) for thermocouple and (2000°C to -1200°C) for resistance thermometers],
- Thermocouples can be used to measure temperatures at a point.
- Thermocouples can be used to measure temperatures at a point.

Advantages of thermocouple thermometers over constant volume gas thermometers

- Thermocouples have a wider temperature range *[(-200° C* to -1600°C) for thermocouple and (270° to 1500°C) for constant volume gas thermometer]
- Thermocouples can be used to measure temperatures at a point,
- Thermocouples can be used to measure temperatures at a point,
- Thermocouples are easy to construct hence cheap.

WORKED EXAMPLES

1. The mercury length is 5cm at ice point and 20cm at steam point. What is the temperature if the mercury length is 8cm.

Solution

$l_0 = 5cm,\ l_{100} = 20cm,\ l_\theta = 8cm\ ,\ \theta = ?$

$$\theta = \left[\frac{l_\theta - l_0}{l_{100} - l_0}\right] \times 100°C$$

$$\theta = \left[\frac{8-5}{20-5}\right] \times 100°C = 20^0C$$

2. The pressure of a gas at constant volume gas thermometer at Kelvin temperature T is $4.8 \times 10^4 Nm^{-2}$. Calculate the temperature T if the pressure at triple point of water is $4.2 \times 10^4 Nm^{-2}$.

Solution

$P_T = 4.8 \text{ X } 10^4\text{Nm}^{-2}$, $P_{tr} = 4.2 \text{ x } 10^4\text{Nm}^{-2}$, T= ?

$$\theta = \left[\frac{P_T}{P_{tr}}\right] \times 273.16K$$

$$\theta = \left[\frac{4.8\times10^4}{4.2\times10^4}\right] \times 273.16K \ = 312.2\text{K}$$

3. The resistance of a certain platinum resistance thermometer is found to be 2.56Ω at 0°C, 3.56 Ω at 100°C, and 6. 78 Ω at 444.5°C; the boiling point of sulphur on the gas scale.

(i) Calculate the boiling point of sulphur on the platinum resistance thermometer.

(ii) The thermometer is immersed in a given liquid and its resistance is observed to be 5.05Ω, Determine the temperature of the liquid on the platinum resistance thermometer.

Solution

(i). $R_0 = 2.56\Omega$, $R_{100} = 3.56\Omega R_{444.5} = 6.78\Omega$, $\theta = ?$

$$\theta = \left[\frac{R_{444.5}-R_0}{R_{100}-R_0}\right] \times 100°\text{C}$$

$$\theta = \left[\frac{6.78-2.56}{3.56-2.56}\right] \times 100°\text{C} = 422^0\text{C}$$

(ii) $R_0 = 2.56\Omega$, $R_{100} = 3.56\Omega$, $R_\theta = 5.05\Omega$, $\theta = ?$

$$\theta = \left[\frac{R_\theta-R_0}{R_{100}-R_0}\right] \times 100°\text{C}$$

$$\theta = \left[\frac{5.05-2.56}{3.56-2.56}\right] \times 100 = 249^0\text{C}$$

4. The resistance of a platinum wire on the platinum resistance thermometer is 2.00 Ω at ice point and 2.73 Ω at steam point. What is the temperature when the resistance of the wire is 8.43 Ω?

Solution

$R_0 = 2\Omega$, $R_{100} = 2.73\Omega$, $R\,\theta = 8.43\Omega$, $\theta = ?$

$$\theta = \left[\frac{R_\theta-R_0}{R_{100}-R_0}\right] \times 100°\text{C}$$

$$\theta = \left[\frac{8.43-2}{2.73-2}\right] \times 100°\text{C} \ = 880.8^0\text{C}$$

5. The table shows the values of resistance and pressure read from a platinum resistance thermometer and constant volume gas thermometer respectively; at steam point, ice point, and at an unknown temperature θ.

Type of thermometer	Steam point	Ice point	Unknown Temperature θ^0c
Platinum resistance thermometer	75 Ω	63 Ω	64.992Ω
Constant volume gas thermometer	$1.1 \times 10^5 Nm^{-2}$	$8 \times 10^4 Nm^{-2}$	$8.51 \times 10^4 Nm^{-2}$

Determine the value of θ on the platinum resistance thermometer and constant volume gas thermometer and account for the difference in the value of θ.

Solution:

For the platinum resistance thermometer

$R_o = 63\Omega$, $R_{100} = 75\Omega$, $R_\theta = 64.992\Omega$, $\theta = ?$

$$\theta = \left[\frac{R_\theta - R_0}{R_{100} - R_0}\right] \times 100°C$$

$$\theta = \left[\frac{64.992 - 63}{75 - 63}\right] \times 100°C = 16.6^0C$$

For the constant volume gas thermometer

$P_o = 8 \times 10^4\ Nm^{-2}$, $P_{100} = 1.1 \times 10^5\ Nm^{-2}$, $P_\theta = 8.51 \times 10^4\ Nm^{-2}$, $\theta = ?$

$$\theta = \left[\frac{P_\theta - P_0}{P_{100} - P_0}\right] \times 100°C$$

$$\theta = \left[\frac{8.51\times10^4 - 8\times10^4}{8.5\times10^4 - 8\times10^4}\right] \times 100 = 17.0^0C$$

The difference in the value of θ for platinum resistance thermometer and constant volume gas thermometer is because they have different thermometric properties which vary differently with temperature.

6. A resistance R_θ of a platinum resistance thermometer is given by

$R_\theta = R_0(1 + a\theta + b\theta^2)$ *where* $a = 1.3 \times 10^{-2}K^{-1}$, $b = 1.33 \times 10^{-6}\ K^{-1}$

R_0 is the resistance at 0°C. Calculate the temperature of the thermometer when the temperature on the constant volume gas thermometer is 300°C.

Solution

From, $R_\theta = R_o(1 + a\theta + b\theta^2)$

$$R_\theta = R_0[1 + (1.3 \times 10^{-2})\theta + (1.33 \times 10^{-6})\theta^2]$$

$$R_{100} = R_0[1 + (1.3 \times 10^{-2}) \times 100 + (1.33 \times 10^{-6}) \times 100^2]$$

$$R_{100} = 2.313R_0$$

$$R_{300} = R_0[1 + (1.3 \times 10^{-2}) \times 300 + (1.33 \times 10^{-6}) \times 300^2]$$

$$R_{300} = 5.02R_0$$

$$\theta = \left[\frac{R_{300} - R_0}{R_{100} - R_0}\right] \times 100°\text{C}$$

$$\theta = \left[\frac{5.02R_0 - R_0}{2.313R_0 - R_0}\right] \times 100 = \left[\frac{R_0(5.02 - 1)}{R_0(2.313 - 1)}\right] \times 100$$

$$\theta = 306.2^0\text{C}$$

TRIAL QUESTIONS

1. The resistance of a platinum resistance wire is 2Ω at the ice point and 2.73Ω at steam point. What temperature on this thermometer corresponds to a resistance of 8.43Ω?

   [Ans: 881^0C]

2. A constant mass of a gas at constant pressure has volume of 200cm^3 at a temperature of pure melting ice and 273.2cm^3 at the temperature of boiling water at standard pressure. Calculate the temperature which corresponds to 525.1cm^3 in the same thermometer.

   [Ans: 444.13K]

3. The pressure recorded by a constant volume gas thermometer at Kelvin temperature T is 4.8 x 10^{-4} Nm^{-2}. Calculate T if the pressure at triple point of water is $4.24\text{x}10^4$ Nm^{-2}

   [Ans: 312.18K]

4. The resistance R_θ of platinum wire at temperature θ°C measured on gas scale is given by,

$R_\theta = R_o = R_o\,(1 + a\theta + b\theta^2)$ *where* a = 3.8 x 10^{-3} *and* b = -5.6 x 10^{-7}.What temperatures will the platinum thermometer indicate when the temperature on the gas scale is 200°C?

[Ans: 197.01^0C]

5. The resistance R_θ of platinum wire at temperature $\theta°C$ measured on scale of mercury thermometer is given by,

$$R_\theta = R_O\,(1 + a\theta + b\theta^2), where\ a = 4.46\ x\ 10^3\ and\ b = 1.8\ x\ 10^{-6}$$

Find the value of the temperature obtained from the thermometer when mercury thermometer reads 430°C. Comment on the two thermometers. [Ans: 450^0C]

6. The volume V_θ of a fixed mass of mercury at temperature $\theta°C$ measured on a perfect gas scale is given by, $V_\theta = V_o(1 + a\theta + b\theta^2). where\ a = 1.818\ x\ 10^{-4} and = 0.8\ x\ 10^{-8}$

Calculate the temperature expected on the mercury thermometer when the temperature on the gas scale is 40°C. [Ans: 0.399]

CHAPTER 14: CALORIMETRY

HEAT AND HEAT CAPACITIES

Heat capacity (C)

This is the quantity of heat required to raise the temperature of a substance by one Kelvin. S.I unit is joules per Kelvin (JK^{-1})

Specific heat capacity (c)

This is the quantity of heat required to raise the temperature of a 1kg mass of a substance by IK. S.I unit is joules per kilogram per Kelvin ($Jkg^{-1}K^{-1}$)

Therefore the quantity of heat Q gained by a substance is given by

$$Q = C.\Delta\theta = mc.\Delta\theta$$

Where m - mass of the substance, c - specific heat capacity of the substances, C - heat capacity of the substance and $\Delta\theta$ is the change in temperature. Thus, specific heat capacity

$$\boldsymbol{c = \frac{Q}{m.\Delta\theta}}$$

And heat capacity, C = mc

COOLING METHOD TO DETERMINE THE SPECIFIC HEAT CAPACITY OF A SOLID

The empty calorimeter and the stirrer are weighed and their combined mass m_c noted. The solid whose specific capacity is required is also weighed and its mass m is noted. The solid is tied on the string and is first placed in a beaker of boiling water. It's left there for about ten minutes.

The calorimeter is half filled with water, weighed again and the combined mass m_1 noted. The mass of water, m_w in the calorimeter is then calculated from $m_w = [m_1 - m_c]$. The initial temperature θ_1 of water is obtained using the thermometer and noted. The solid is quickly transferred from boiling water to the calorimeter. The water is well stirred and the highest temperature θ_2 reached is noted. Let c_w, c_c *and* c be the specific heat capacity of water, calorimeter and solid respectively. Assuming that the water boils at 100^0C and that, there's negligible heat loss to the surrounding

$$\begin{pmatrix} heat\ lost \\ by\ the\ solid \end{pmatrix} = \begin{pmatrix} heat\ gained \\ by\ water \end{pmatrix} + \begin{pmatrix} heat\ gained\ by \\ the\ calorimeter \end{pmatrix}$$

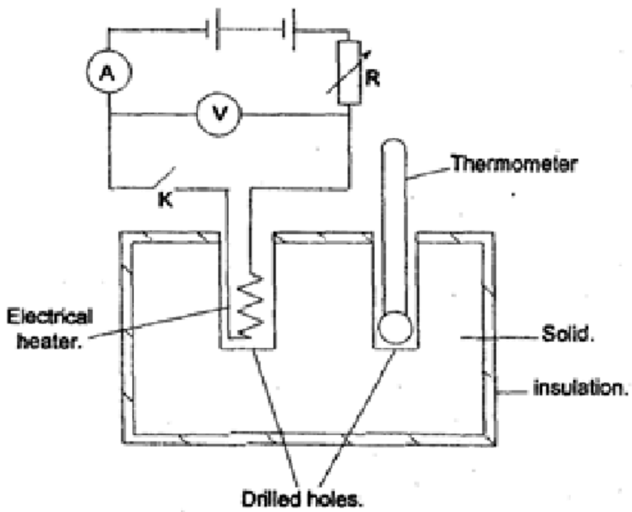
$$m_w\, c_W\,(\theta_2 - \theta_1) + m_c c_c(\theta_2 - \theta_1) = mc(100 - \theta_2)$$

$$(m_w c_w + m_c c_c)\,(\theta_2 - \theta_1) = mc(100 - \theta_2)$$

$$c = \left[\frac{(m_w C_{w+} m_c c_c)(\theta_2 - \theta_1}{m(100 - \theta_2)}\right]$$

ELECTRICAL METHOD TO DETERMINE SPECIFIC HEAT CAPACITY OF SOLID

This method is suitable for good conductors of heat such as metals.



Two holes are drilled into the solid; one for the heater and the other for the thermometer. The solid is weighed and its mass, m is obtained. The solid is then surrounded by an insulator. The heater and the thermometers are placed in their respective holes. Switch K is closed and the ammeter reading, I and voltmeter reading, V are adjusted to reasonable values using a rheostat R. Switch K is then opened and the initial temperature $\boldsymbol{\theta_1}$ noted. Switch K is again closed and the stop clock immediately started. The ammeter and voltmeter readings are noted. After sometime, switch K is opened and the time, t is noted from the stop clock. The highest temperature $\boldsymbol{\theta_2}$ is noted when K is opened.

Assuming there's no heat loss,

$$(Chemical\ energy\ converted\ into\ heat) = (heat\ gained\ by\ the\ solid)$$

$$IVt = mc(\theta_2 - \theta_1)$$

$$c = \left[\frac{IVt}{m(\theta_2 - \theta_1}\right]$$

Where *c* is the specific heat capacity of the solid.

Precautions to reduce heat losses

- There should be thermal contact between the thermometer and the drilled hole. This is achieved by adding a small amount of oil in the hole for the thermometer before inserting the thermometer.
- The solid is covered by felt jacket to reduce heat loss by conduction.
- The metal block is polished to reduce heat loss by radiation.

NOTE: Despite these precautions, some heat is still lost to the surrounding therefore this method is not very accurate. Sources of errors in the above experiment

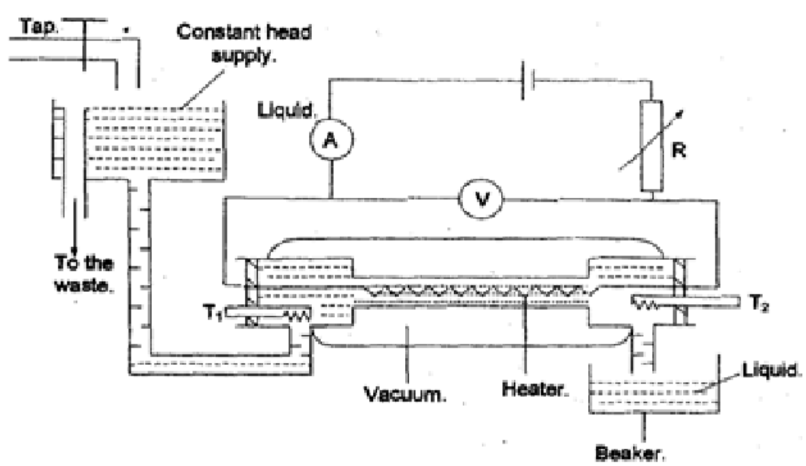
i. Some heat energy is used to raise the temperature of the heater and the thermometer.

ii. The insulator also absorbs some heat. Assumption made

It's assumed that the heat lost to the surrounding is negligible.

CONTINUOUS FLOW METHOD TO DETERMINE THE SPECIFIC HEAT CAPACITY OF A LIQUID

The set up is as below. T_1 and T_2 are thermometers. The liquid whose specific heat capacity is required is continuously passed through the apparatus at a constant rate. The heater heats the liquid until steady state conditions are reached. That is, the inflow temperature θ_1, and outflow temperature θ_2 are constant.



At this moment, no heat is taken into the apparatus and therefore all the heat supplied raises the temperature of the liquid. Temperatures θ_1 and θ_2 are read from the thermometers T_1 and T_2, respectively and noted.

The values of the current I_1 and voltage V_1 in the circuit are read from the ammeter A and voltmeter V respectively and noted. The mass m_1 of the liquid which flows per second is also found. Therefore

$$\begin{pmatrix} power \\ supplied \\ by\ the\ heater \end{pmatrix} = \begin{pmatrix} rate\ of\ heat \\ gained\ by \\ the\ liquid \end{pmatrix} + \begin{pmatrix} power\ lost \\ to\ the \\ surrounding \end{pmatrix}$$

$$I_1V_1 = m_1c(\theta_2 - \theta_1) + H \ldots\ldots\ldots (1)$$

To eliminate the power lost to the surrounding H_1 the experiment is repeated for a different flow rate m_2 at the same steady temperatures θ_1 and θ_2 . The new values of current I_2 and voltage $V_2$are noted. Similarly

$I_2V_2 = m_2c(\theta_2 - \theta_1) + H \ldots\ldots\ldots\ldots\ldots (2)$

Subtracting equation (1) from *(2)* gives

$I_2V_2 - I_1V_1 = (m_2c(\theta_2 - \theta_1) + H) - (m_1c(\theta_2 - \theta_1) + H)$

$$I_2V_2 - I_1V_1 = c(m_2 - m_1)(\theta_2 - \theta_1)$$

$$c = \left[\frac{(I_2V_2 - I_1V_1}{(m_2 - m_1)(\theta_2 - \theta_1)}\right]$$

Where c is the specific heat capacity of the liquid.

Advantages of continuous flow method

- The specific heat capacity of the apparatus is not required since the apparatus doesn't absorb heat at steady state conditions.
- The method is very accurate since errors due to heat lost to the surrounding are eliminated by repeating the experiment.
- The inflow and outflow temperatures are set at steady state conditions and therefore can be measured accurately at leisure.

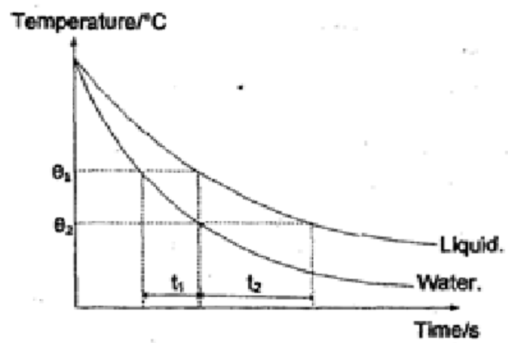
Advantages of continuous flow method

- It requires a test liquid to be available in large quantities.
- It's used for measuring the specific heat capacity of liquids only and not solids.

COOLING METHOD TO DETERMINE THE SPECIFIC HEAT CAPACITY OF A LIQUID

An empty calorimeter is weighed and then filled with a known volume of hot water. The calorimeter is placed on an insulating stand and water is well stirred using a thermometer. The temperature of the hot water is read and noted every half a minute. A graph of temperature against time is plotted. The calorimeter and water are weighed and their masses m_w and m_c

respectively are noted. The water is poured away and the calorimeter is thoroughly dried.
An equal volume of the liquid is heated, poured into the calorimeter and placed on an insulating stand near an open window. The liquid is well stirred using a thermometer and the temperature of the liquid is read and noted every half a minute. The calorimeter and the liquid are weighed and the mass of the liquid m is calculated. A graph of temperature against time is plotted on the same axes.



Let C_w C_c and C be the specific heat capacity of water, calorimeter and solid respectively. Consider a temperature fall from θ_1 to θ_2 in both experiments. By Newton's law of cooling,

$$\binom{rate\ of\ heat\ loss}{with\ temperature\ experiment} = \binom{rate\ of\ heat\ loss}{with\ liquid\ experiment}$$

$$\left[\frac{m_w c_w(\theta_1-\theta_2)+m_c c_c(\theta_1-\theta_2)}{t_1}\right]=\left[\frac{mc(\theta_1-\theta_2)+ m_c c_c(\theta_1-\theta_2)}{t_2}\right]$$

$$\left[\frac{(m_w c_w+m_c c_c)(\theta_1-\theta_2)}{t_1}\right]=\left[\frac{(mc+m_c c_c)}{t_2}\right]$$

$$c=\left[\frac{t_2(m_w c_w+m_c c_c)}{mt_1}\right]-\left(\frac{m_c c_c}{m}\right)$$

Where c is the specific heat capacity of the liquid.

HEAT LOSSES IN CALORIMETRY

Heat losses in calorimetry include:

- Heat loss by radiation,
- Heat loss by convection,
- Heat loss by conduction.

Precautions taken to minimize heat losses in calorimetry

- Using highly polished surfaces to reduce heat loss by radiation.
- Using a lid to minimize heat loss by convection.
- Lagging the calorimeter with an insulating material to reduce heat loss by conduction,
- Surrounding the calorimeter with vacuum to reduce heat loss by convection and conduction.

NOTE: Despite all these precautions, heat losses can't be eliminated because even the best insulator has some conductivity. Therefore, for great accuracy, a cooling correction is made to cater for the heat lost to the surrounding during the heating process.

COOLING CORRECTION

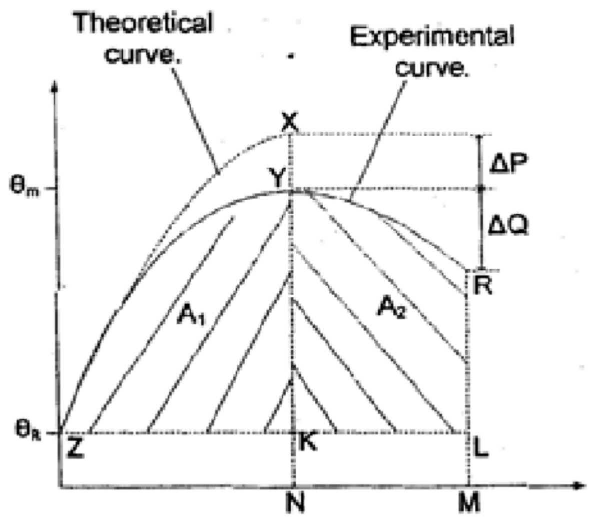
Cooling correction is defined as the number of degrees Celsius that should be added to the observed maximum temperature of mixtures to cater for the for heat losses during temperature rise. Cooling correction can also be defined as the extra temperature difference to be added to the observed maximum temperature of the mixture to make up for the heat losses to the surrounding during heating.

Estimating cooling correction in determination of specific heat capacity of a poor conductor of heat by method of mixtures

A substance is heated in boiling water for about 10 minutes and then quickly transferred to cold water in a calorimeter and the stop clock started.

Water is well stirred and the temperature is recorded after every half a minute. After attaining the highest temperature, continue recording for a time equal to that taken to reach the highest temperature.

Suppose that in an experiment, the temperature against time graph looks like the one shown in figure that follows with the experimental curve containing observed values and theoretical curve containing value that would have been obtained without heat losses.



A vertical line YN is drawn through the peak Y. ensuring that ON≈NM, another vertical line RM is drawn. A horizontal line ZKL is drawn through the room temperature θ_R.

Area A_1 of ZKY and area A_2 of YKLR are found by counting small squares on the graph paper. Therefore

$$\frac{\Delta P}{\Delta Q} = \frac{A_1}{A_2}$$

$$\Delta P = \frac{A_1}{A2} \times \Delta Q$$

ΔP is the cooling correction.

NEWTON'S LAW OF COOLING

Newton's law of cooling states that the rate of heat loss by a body is directly proportional to its excess temperature over that of the surrounding under conditions of forced convections of air. That is

$$\frac{dQ}{dt} \propto (\theta - \theta_R)$$

Where θ is the temperature of the body at a time *t.*

θ_R *is* the room temperature (temperature of the surrounding)

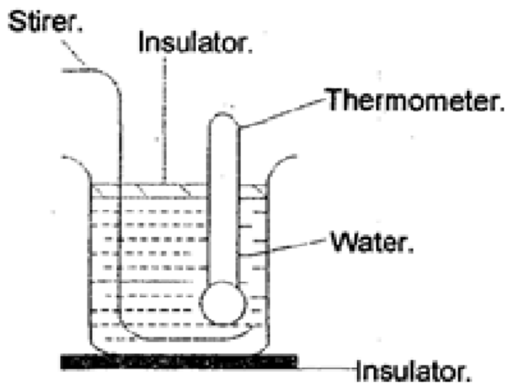
Q is the quantity of heat.

If k is the constant of proportionality, then

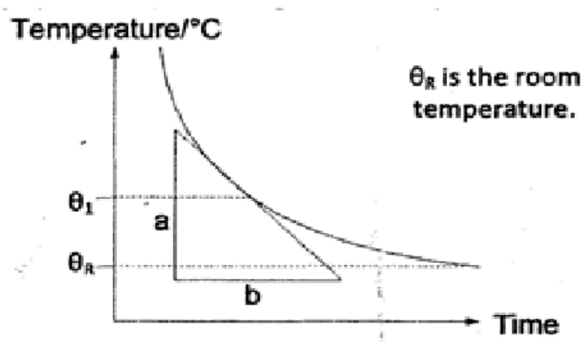
$$\frac{dQ}{dt} = -k\,(\theta - \theta_R)$$

k depends on the material of the body and it's only true for a large excess temperature $(\theta - \theta_R)$. The negative sign is introduced in the equation since temperature is falling and heat is being lost.

Experiment to verify Newton's law of cooling



Water in the calorimeter is heated to boiling point. The calorimeter is placed on an insulator and then subjected to a stream of air from a nearby window such that the water begins to cool. The temperature of water is recorded at suitable time intervals and a graph of temperature against time is plotted.

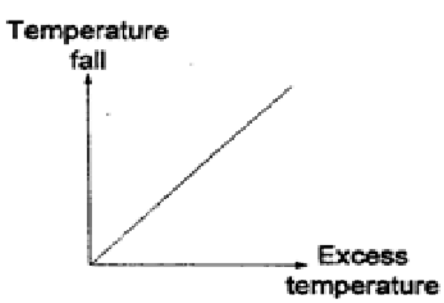


At different temperatures θ_1 the slope of the graph is determined and the corresponding excess temperature calculated. That is, at a temperature θ_1

$$(slope) = \left(\frac{a}{b}\right) = (rate\ of\ temperature\ fall), and$$

$$(excess\ temperature) = (\theta_1 - \theta_R)$$

A graph of temperature fall against excess temperature is plotted. A straight line is obtained implying that the rate of temperature fall is proportional to excess temperature which verifies Newton's law of cooling.



LATENT HEAT AND SPECIFIC LATENT HEAT

Latent heat

The heat that a body absorbs in melting, evaporating or sublimating and gives out in freezing or condensing is called latent heat. This is because it doesn't cause change in temperature of the body despite causing the change in state. S.I unit is joules (*J*)

Therefore, the latent heat Q gained of a body is given by

$$Q = ml$$

Where m = mass of the substance, l = specific latent heat of the body

Specific latent heat of fusion

This is the quantity of heat required to change a 1kg mass of a substance from solid state to liquid state without change in temperature.

Specific latent heat of vaporization

This is the quantity of heat required to change a 1kg mass of a substance from liquid state to gaseous state without change in temperature.

EXPERIMENT TO DETERMINE THE SPECIFIC LATENT HEAT OF FUSION OF ICE BY METHOD OF MIXTURES

The empty calorimeter and stirrer are weighed. Let m_c be the mass of the calorimeter and stirrer. Water is poured into the calorimeter and the calorimeter is weighed again with water. Let m_1 be the combined mass of calorimeter and water. Therefore, the mass of water poured into the calorimeter is given by $m_w = [m_1 - m_c]$. The water is warmed till the temperature is about 5°C above room temperature. Let θ_1 be the temperature of the calorimeter and water. Small pieces of ice are taken, dried between the filter paper and dropped into the water in the calorimeter. The water is well stirred till the ice has melted. More pieces of ice are added and the water is well stirred till the temperature of the mixture is 5°C below room temperature. Let θ_2 be the temperature of the mixture. The calorimeter and water are weighed again and the mass of ice is obtained. Let m_2 be the new mass of water and calorimeter. Then, mass of ice, $m = (m_2 - m_1)$. Let c_w and c_c be the specific heat capacity of water and calorimeter respectively. By conservation of energy,

$$\begin{pmatrix} heat\ gained \\ by \\ melting\ ice \end{pmatrix} + \begin{pmatrix} heat\ gained \\ by \\ molten\ water \end{pmatrix} = \begin{pmatrix} heat\ lost \\ by\ water \\ and\ calorimeter \end{pmatrix}$$

$$ml_f + mc_w(\theta_2 - 0) = (m_w c_w + m_c c_c)(\theta_1 - \theta_2)$$

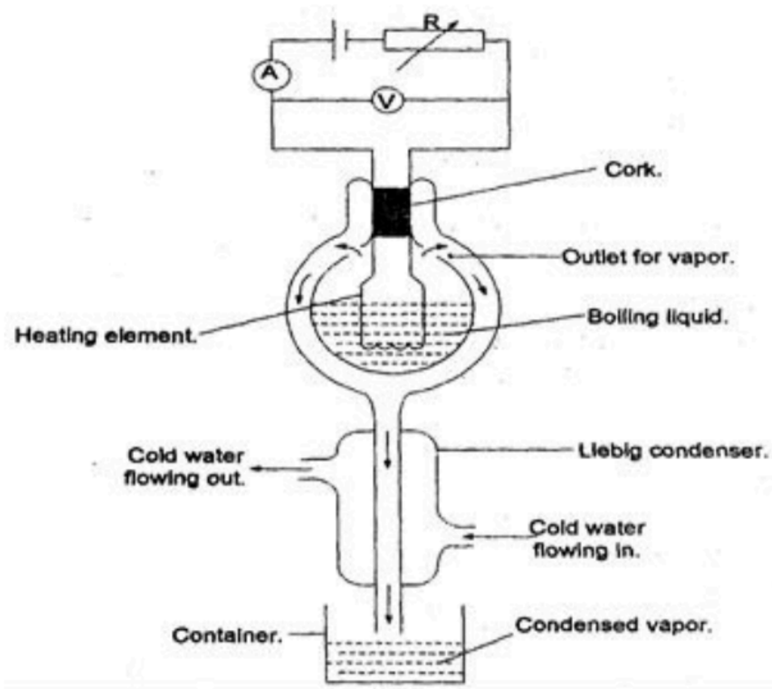
$$l_f = \left[\frac{(m_w c_w + m_c c_c)(\theta_1 - \theta_2) - mc_w\theta_2}{m}\right]$$

Where l_f is the specific latent heat of fusion of ice.

Advantage of this method

Cooling correction is not required since water is first heated 5°C above room temperature and then cooled 5°C below room temperature.

ELECTRICAL METHOD TO DETERMINE THE SPECIFIC LATENT HEAT OF VAPORIZATION OF A LIQUID



When the circuit is complete, the heater will heat the liquid until steady state conditions are reached. At steady state, all the heat supplied is used to vaporize the liquid and none is used to heat the apparatus.

Current I and voltage V are noted and the mass m_1 of the condensed vapour in time, t is determined. The mass m of condensed vapour per second is obtained from, $m = \frac{m_1}{t}$

By conservation of energy,

$$\begin{pmatrix} power\ lost \\ by\ the \\ heater \end{pmatrix} = \begin{pmatrix} heat\ gained\ to \\ vapourise\ the\ liquid \\ per\ second \end{pmatrix} + \begin{pmatrix} power\ lost \\ to\ the \\ surrounding \end{pmatrix}$$

$$IV = ml + H \ldots\ldots\ldots (1)$$

The procedure is repeated for different values of *I and V* and the analysis made as above. That is,

$$I'V' = M'l + H \ldots\ldots\ldots(2)$$

Subtracting equation (2) from (1) gives;

$$IV - I'V = (ml + W) - (m'l + H)$$

$$(IV - I'V') = (m - m')l$$

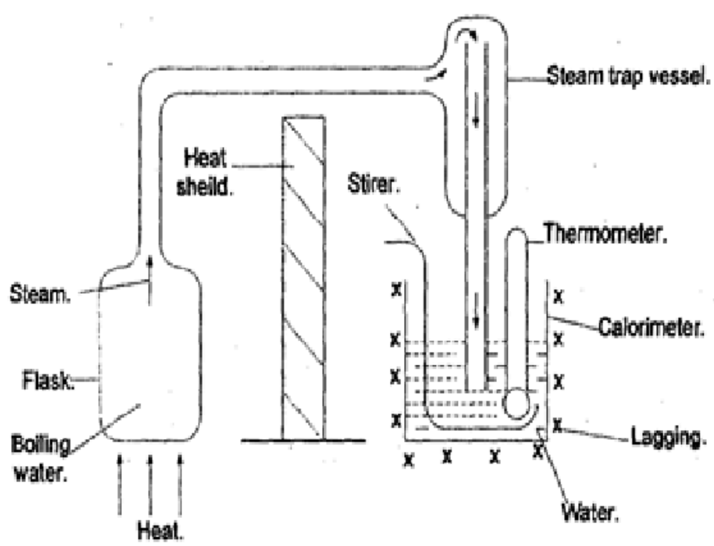
$$l = \left[\frac{IV - I'V'}{m - m'}\right]$$

l is the specific latent heat of vaporization of the liquid.

Advantages of this method

- It's a very accurate method.
- Heat loss to the surrounding is eliminated.
- The heat capacity of the apparatus is not required.

EXPERIMENT TO DETERMINE THE SPECIFIC LATENT HEAT OF VAPORIZATION OF WATER BY METHOD OF MIXTURES



The empty calorimeter and stirrer are weighed. Let m_c be the mass of the calorimeter and stirrer. Some water is poured into the calorimeter and the calorimeter is weighed again with water. Let m_1 be the combined mass of calorimeter and water. Therefore, the mass of water poured into the calorimeter is given by $m_w = [m_1 - m_c]$.

The calorimeter is surrounded by a poor conductor of heat. The temperature θ_1 of the calorimeter and water is determined.

Water in the flask is heated till it starts boiling. The flask is then attached to the steam trap and the outlet of the steam trap is dipped in water in a calorimeter. The steam condenses in the water and the water is well stirred. After sometime, the steam trap is removed from the water in the calorimeter. The water is well stirred the highest temperature θ_2 of the mixture is noted. The calorimeter is removed from the poor conductor and reweighed to find the mass of steam. Let m_2 be the new mass of water and calorimeter. Then, mass of condensed steam, $m = (m_2 - m_1)$. Let c_w *and* c_c be the specific heat capacity of water and calorimeter respectively. By conservation of energy,

$$\begin{pmatrix} \text{heat lost} \\ \text{by} \\ \text{condensing steam} \end{pmatrix} + \begin{pmatrix} \text{heat lost} \\ \text{by} \\ \text{condensed water} \end{pmatrix} = \begin{pmatrix} \text{heat gained} \\ \text{by water} \\ \text{and calorimeter} \end{pmatrix}$$

$$ml + mc_w(100 - \theta_2) = (m_w c_w + m_c c_c)(\theta_2 - \theta_1)$$

$$l = \left[\frac{(m_w c_w + m_c c_c)(\theta_2 - \theta_1) - mc_w\theta_2}{m}\right]$$

Where *l* is the specific latent heat of vaporization of water.

Assumption made

It's assumed that water boils at 100°C.

Source of error

Some steam will condense before reaching the calorimeter.

KINETIC THEORY OF LATENT HEAT

Why temperature remains constant during change of phase

During change of phase, energy supplied goes into increasing amplitude of oscillation of the atoms. The amplitude of oscillation becomes so large that the regular arrangement of the atoms collapses. Until the process is complete, the temperature remains constant.

Significance of latent heat in regulation of body temperature

On a hot day, when the temperature is high, a layer of moisture (sweat) is formed over the body. Evaporation takes place at the surface of the body and the temperature of the sweat falls. The latent heat absorbed by the sweat as it evaporates is taken from the body so reducing the body's temperature.

Why specific latent heat of vaporization is regarded as a molecular potential

During taking in of specific latent heat of vaporization, there's no temperature rise of the surrounding. Therefore, all the heat is used to overcome the intermolecular forces of attraction of the liquid to form the vapour where molecules are farther apart. This energy supplied to break the intermolecular forces is the potential energy for the liquid hence specific latent heat of vaporization is regarded as a molecular potential.

Why specific latent heat of vaporization of water is higher at 20°C than at its boiling point (100°C)

The intermolecular forces of attraction between water molecules are higher at 20°C than at its boiling point. This is because at boiling point, water molecules have a higher mean speed and spend less time in the vicinity of each other. Therefore, the energy required to overcome attractive forces between molecules at 20°C is greater than at its boiling point.

Why specific latent heat of vaporization has a greater value than that of fusion

At melting, specific latent heat of fusion is taken in by the solid to break the intermolecular forces holding the molecules together in their fixed positions and change into liquid state. The molecules are still closer together and experience stronger intermolecular forces of attraction such that specific latent heat of fusion is relatively small.

However, for vaporization to occur, a large amount of energy is still required to separate the liquid molecules and allow them to move around independently. Furthermore, some energy is still required to enable the vapor to expand against the atmospheric pressure. The energy for these two operations is supplied as specific latent heat of vaporization.

Thus, specific latent heat of vaporization has a greater value than specific latent heat of fusion.

WORKED EXAMPLES

Where applicable, use the following constants

Specific heat capacity of water = $4200 Jkg^{-1} K^{-1}$.

Specific latent heat of fission of ice = $3.36 \times 10^5 Jkg^{-1}$.

Specific heat capacity of copper = $400kg^{-1}k^{-1}$.

1. A solid of mass 0.5kg and specific heart capacity of $4 \times 10^{2} Jkg^{-1}K^{-1}$ at a temperature of 90°C is placed into a mixture of ice and 0.1kg of water contained in a vacuum flask. The final temperature of the mixture is found to be 10°C. Calculate the mass of ice immersed in water.

Solution

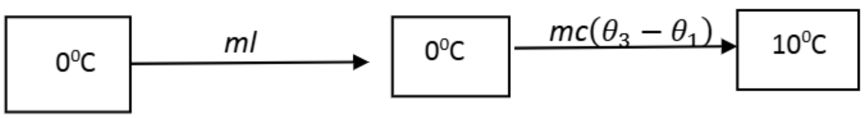
Let m be the mass of ice immersed in water

$\theta_1 = 0^oC, \theta_2 = 90^oC, \theta_3 = 10^oc, m_w = 0.1kg, c_s = 4x10^2Jkg^{-1}k^{-1}$

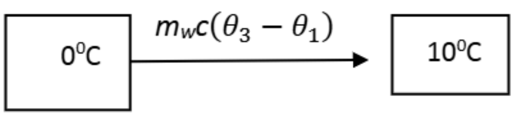
$c = 4200Jkg^{-1}k^{-1}, l = 3.36x10^5Jk^{-1}$

heat gain is illustrated as below;

For ice;



For water;



Total heat gained is given by;

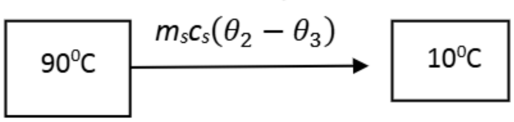
$Q_{gained} = ml + mc(\theta_3 - \theta_1) + m_wc(\theta_3 - \theta_1)$

$= (3.36 \times 10^5)m + 4200(10 - 0)m + 0.1 \times 4200(10 - 0)$

$Q_{gained} = (3.78 \times 10^5)m + 4200$ ..... (i)

Heat gain is illustrated as below

For solid;



Total heat lost is given by, $Q_{lost} = m_sc_s(\theta_2 - \theta_3)$

$= 0.5 \times 4 \times 10^2 \times (90 - 10) = 1.6 \times 10^4J$

By conservation of energy, $Q_{gained} = Q_{lost}$

$(3.78x10^5)m + 4200 = 1.6x10^4$

$m = \frac{1.6\times10^4 - 4200}{3.78\times10^5} = 3.122 \times 10^{-2}kg$

2. In an experiment to determine the specific latent heat of vaporization, a current of 1A was flowing per second with a p.d of 5V and the mass of the liquid collected in 100s was 0.8g. When

the p.d was increased to 5.4V, the current was 2A and a mass of 3.6g of liquid was collected in 100s. Calculate:

(i) The specific latent heat of vaporization of the liquid,

(ii)The rate of loss of heat to the surrounding.

Solution

Let l be the specific latent heat of vaporization of the liquid, and H be the rate of loss of heat to the surrounding.

Considering the first experiment, $I_1 = 1A$, $V_1 = 5V$

Mass of liquid collected per second

$$m_1 = \frac{0.8 \times 10^{-3}}{100} = 8 \times 10^{-6} kg/s$$

By conservation

$$\begin{pmatrix} \text{power lost} \\ \text{by the} \\ \text{heater} \end{pmatrix} = \begin{pmatrix} \text{heat gained to} \\ \text{vapourise the} \\ \text{liquid per second} \end{pmatrix} + \begin{pmatrix} \text{rate of heat} \\ \text{lost to} \\ \text{the surrounding} \end{pmatrix}$$

$$I_1 V_1 = m_1 l + H$$

$$1 \times 5 = (8 \times 10^{-6})l + H$$

$$H = 5 - (8x10^{-6})l \dots\dots\dots \quad (i)$$

Considering the second experiment, $I_2 = 2A$, $V_2 = 5.4V$

Mass of liquid collected per second

$$m_2 = \frac{3.6x10^{-3}}{100} = 3.6x10^{-5} kg/s$$

By conservation, $I_2 V_2 = m_2 l + H$

$$2 \text{ x } 5.4 = (3.6 \text{ x } 10^{-5})\, l + H$$

$$H = 10.8 - (3.6 \text{ x } 10^{-5})\, l$$

Equating *(i) and (ii)* gives;

$$5 - (8 \times 10^{-6})l = 10.8 - (3.6 \times 10^{-5})l$$

$$[(3.6 \times 10^{-5}) - (8 \times 10^{-6})]l = 10.8 - 5$$

$$l = 2.071 \times 10^{5} Jkg^{-1}$$

(ii) The rate of loss of heat to the surrounding

$$H = 10.8 - (3.6 \times 10^{-5})l$$

$$H = 10.8 - (3.6 \times 10^{-5})(2.071 \times 10^{5})$$

$$H = 3.344W$$

3. In a continuous flow experiment, $3.6 \times 10^{-3} m^3$ of the liquid flows through the apparatus in 10 minutes. When electrical power is supplied to the heating coil at rate of 44W, a steady difference of 4K is obtained between the temperature of the out-flowing and the in-flowing liquid. When the flow rate is increased to $4.8 \times 10^{-3} m^3$ of the liquid in l0 minutes, the electrical power required to maintain the temperature difference is 58W. Find the

i. Specific heat capacity of the liquid,

ii. Rate of loss of heat to the surrounding, (consider the density of the liquid to be $800kgm^{-3}$)

Solution

$\rho = 800kgm^{-3}$, $t = 10 \times 60 = 600s$ considering the first experiment

$P_1 = 44W, \Delta\theta = 4K, v_1 = 3.6 \times 10^{-3}m^3$

Mass of liquid collected per second

$$m_1 = \frac{v_1\rho}{t} = \frac{3.6 \times 10^{-3} \times 800}{600} = 4.8 \times 10^{-3} kg/s$$

$$\begin{pmatrix} power \\ supplied\ by \\ the\ heater \end{pmatrix} = \begin{pmatrix} rate\ of\ heat \\ gained\ by \\ the\ liquid \end{pmatrix} + \begin{pmatrix} rate\ of\ heat \\ lost\ to\ the \\ surrounding \end{pmatrix}$$

$$44 = 4 \times 4.8 \times 10^3 \times 10c + H$$

$$H = 44 - 0.192c \qquad (i)$$

Considering the second experiment

$P_2 = 58W, \Delta\theta = 4K, v_2 = 4.8x10^{-3}m^3$

Mass of liquid collected per second

$$m_2 = \frac{v_2\rho}{t} = \frac{4.8 \times 10^{-3} \times 800}{600} = 6.4 \times 10^{-3} kg/s$$

$$P_2 = m_2 c\Delta\theta + H$$

$$58 = 4 \times 6.4 \times 10^{-3} \times 10c + H$$

$$H = 58 - 0.256c \qquad (ii)$$

Equating (i) and (ii) gives;

$$44 - 0.192c = 58 - 0.256c$$

$$0.064c = 4$$

$$c = 218.8Jkg^{-1}K^{-1}$$

(ii) Rate of loss of heat to the surrounding

$$H = 58 - 0.256C$$

$$H = 58 - 0.256 \times 218.8$$

$$H = 2W$$

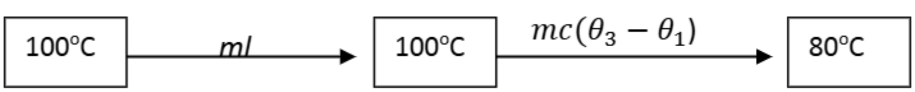
4. A 600W electric heater is used to raise the temperature of a certain mass of water from room temperature to 80°C. Alternatively, by passing steam from the boiler into the same initial mass of water at the same initial temperature, the same temperature rise is obtained in the same time. If 16g of steam was being evaporated every minute in a boiler, find the specific latent heat of vaporization of steam assuming that there were no heat losses.

Solution

Considering the boiler, mass of steam evaporated from the boiler per second

$$m = \frac{16 \times 10^{-3}}{60} = 2.667 \times 10^{-4} kg/s$$

For steam



Total rate of heat gained when using steam from the boiler is given by,

$$P_{steam} = ml + mc(\theta_3 - \theta_1)$$

$$= (2.667 \times 10^{-4})l + 2.667 \times 10^{-4} \times 4200 \times (100 - 80)$$

$$P_{steam} = (2.667 \times 10^{-4})l + 22.4 \quad \ldots\ldots\ldots\ldots\ldots \quad (i)$$

Considering the heater, total rate of heat gained when using the electrical heater is given by

$$P_{heater} = 600W$$

Since the temperature rise is the same, then, $P_{steam} = P_{heater}$

$$(2.667 \times 10^{-4})l + 22.4 = 600$$

$$(2.667 \times 10^{-4})l = 577.6$$

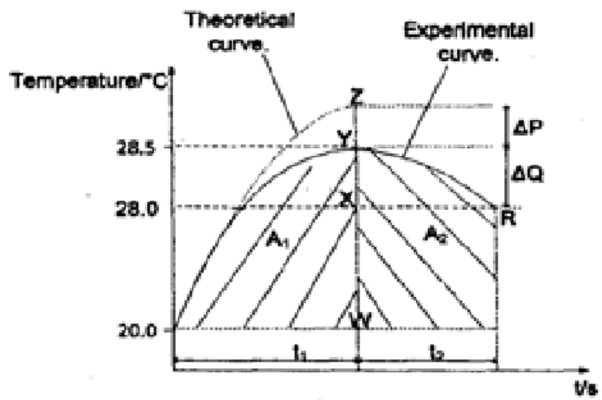
$$l = 2.166 \times 10^{6} Jkg^{-1}$$

5. A copper calorimeter of mass 50g contains 100g of a certain liquid. The initial temperature is 20°C. A heater of negligible heat capacity is immersed in a liquid and operated at 1.5A, 7.5V for exactly 5minutes. After this time, the temperature is recorded as 28.5°C. Subsequently, the temperature falls steadily to reach 28°C after 2 minutes. Calculate:

(i) The corrected temperature rise.

(ii) The specific heat capacity of the liquid.

Solution



After 5 minutes, t_1 = 5 x 60 = 300s

$$A_1 = \frac{1}{2} \times t_1 \times WY = \frac{1}{2} \times 300 \times (28.5 - 20.0) = 1275$$

2minutes after the subsequent fall in temperature,

$$t_2 = 2 \text{ x } 60 = 120\text{s}$$

$$A_2 = \left[\left(\frac{1}{2} \times t_2 \times XY\right) + (t_2 \times WX)\right]$$

$$A_2 = \frac{1}{2} \times 120 \times (28.5 - 28.0) + 120 \times (28.0 - 20.0) = 990$$

The corrected temperature rise, $\Delta P = \frac{A_1}{A_2} \times \Delta Q$

$$\Delta P = \frac{1275}{990} \times (28.5 - 28.0) = 0.6439^oC$$

(ii) $1 = 1.5A, V = 7.5V, \theta = 20^OC, m = 0.1kg, m_c = 0.05, c_c = 400Jkg^{-1}k^{-1}$

$$\begin{pmatrix} Corrected \\ maximum \\ temperature \end{pmatrix} = \begin{pmatrix} observed \\ maximum \\ temperature \end{pmatrix} + \begin{pmatrix} corrected \\ temperature\ rise \end{pmatrix}$$

$$\theta_{max} = 28.5 + 0.6439 = 29.1439^0\text{C}$$

Let *c* be the specific heat capacity of the liquid. By conservation of energy,

$$\begin{pmatrix} heat\ lost \\ by\ heater \end{pmatrix} = \begin{pmatrix} heat\ gained \\ by\ the\ liquid \end{pmatrix} + \begin{pmatrix} heat\ gained\ by\ the \\ copper\ carolimeter \end{pmatrix}$$

$$IVt_1 = mc(\theta_{max} - \theta) + m_c c_c(\theta_{max} - \theta)$$

$$1.5 \times 7.5 \times 300 = 0.1(29.1439 - 20)c + 0.05 \times 400(29.1439 - 20)$$

$$0.191439c = 3192$$

$$c = 3491Jkg^{-1}K^{-1}$$

Trial Questions:

1. A body of mass 0.2kg at 100°C is dropped into 0.08kg of water at 15°C contained in a calorimeter of mass 0.12kg having specific heat capacity of 400$Jkg^{-1}K^{-1}$. Given that the final steady temperature reached is 35°C. Calculate the specific heat capacity of the body. State any assumption made. Specific heat capacity of water = 4200 $Jkg^{-1}K^{-1}$ [Ans: 590.77 $Jkg^{-1}K^{-1}$]
2. 0.2kg of iron at 100°C is dropped into 0.09kg of water at 20°C inside a calorimeter of mass 0.15kg and specific heat capacity of 800 $Jkg^{-1}K^{-1}$. Find the final temperature of iron assuming heat losses are negligible. [Ans: 37.3K]
3. 3.21g of a liquid at 60°C is mixed with 100g of water at 12.5^{0}C which is already in the calorimeter of mass 70g and specific heat capacity of 400$Jkg^{-1}K^{-1}$. Find the new temperature of water assuming heat losses are negligible. Specific heat capacity of liquid = 400$Jkg^{-1}K^{-1}$.
   [Ans: 12.64^{0}C]
4. 50g of ice at 0°C is added to 200g of water initially at 70°C in a vacuum flask. When all the ice has melted, the temperature of the flask and its components drops to 40°C. On adding a further 80g of ice, the temperature of the whole becomes 10°C when all the ice has melted. Calculate the specific latent heat effusion of ice. [Ans: $3.78 \times 10^{-5} Jkg^{-1}$]
5. A liquid flows at a rate of 0.15kg per minute through a tube and is heated by a heater dissipating 25.2W of power. The inflow and outflow temperatures of the liquid are 15.2°C and 17.4°C respectively. When the rate of flow is increased to 0.2318kg per minute and power adjusted to 37.8W.Tthe inflow and outflow temperatures are not altered. Find
   (i) Specific heat capacity of the liquid,
   (ii) The rate of loss of heat to the surrounding [Ans: 420.09 $Jkg^{-1}K^{-1}$]
6. In the determination of specific heat capacity of a liquid using continuous flow method, the following readings were obtained; e.m.f = 3.05V, current = 6.55A and 0.431kg of the liquid passes through the apparatus in 20minutes. Second experiment: e.m.f = 3.15V, current = 7.54A and 0.524kg of water pass through the apparatus in 20minutes giving the same temperature rise as before. Use the results above to calculate the specific heat capacity of the liquid and power lost to the surrounding. [Ans: 2420 $Jkg^{-1}K^{-1}$, 2.62W]
7. In a continuous flow experiment, a steady difference of temperature of 15°C is maintained when the rate of liquid flow is 45g per second and the rate of electrical heating is 60.5W. On reducing the liquid flow rate to 15g per second, 36.5W is required to maintain the same temperature. Calculate the

(i) Specific heat capacity of the liquid,

(ii) Rate of heat loss to the surrounding. [Ans: 53.3 $Jkg^{-1}K^{-1}$, 24.52W]

8. When electrical power is supplied at a rate of 12.0W to a boiling liquid, 8.6 x 10^{-3}kg of the liquid evaporates in 30 minutes. On reducing the power to 7.0W, 5 x 10^{-3}kg of the liquid evaporates in the same time.

(i) Calculate the specific latent heat of vaporization of the liquid,

(ii) Power lost to the surrounding. [Ans: $2.5 \times 10^6 Jkg^{-1}, 0.05W$]

CHAPTER 15: THERMAL CONDUCTION

Conduction is the flow of heat through matter from hotter regions to colder regions without movement of matter as a whole.

MECHANISM OF THERMAL CONDUCTION

(a). In metals (good conductors)

When a metal is heated at one end, the atoms at the heated end gain kinetic energy and vibrate with greater amplitude. This vibration energy is transmitted through the atomic bonds to the neighboring atoms forming a wave that transmits energy from the end of the metal to the other end.

The electrons at the heated end also gain energy and move faster. Since these electrons are delocalized, they move from one end of the metal to the other transmitting energy to the rest of the metal.

(b). In non-metals (poor conductors)

When a non-metal is heated at one end, the atoms at the heated end gain kinetic energy and vibrate with greater amplitude. This vibrational energy is transmitted through the atomic bond to the neighboring atoms forming a wave that transmits energy from one end of the non-metal to the other end.

Why metals are better conductors than non-metals

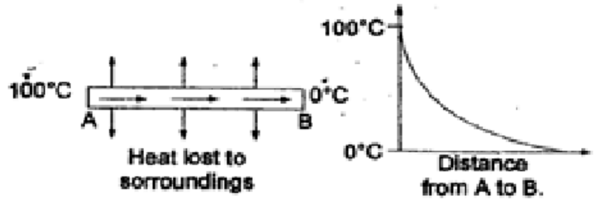
In metals, heat is transmitted movement of free electrons and by inter-atomic vibrations. In non-metals, heat is transmitted only through inter-atomic vibrations since non-metals don't have free electrons.

Since electrons are very small and highly mobile, heat transfer due to motion of electrons is faster than heat transfer due to interatomic vibrations. Therefore, metals are better conductors of heat than non-metals.

TEMPERATURE DISTRIBUTION ALONG A CONDUCTOR

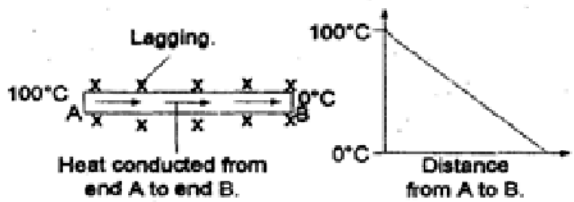
(a). Along an unlagged good conductor

The temperature distribution is not uniform along the conductor because of the heat lost to the surrounding.

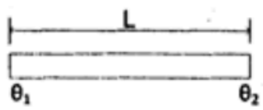


(b) Along a well lagged good conductor

The temperature distribution is uniform along the conductor because no heat lost to the surrounding.



Temperature gradient



If $\theta_2 > \theta_1$ then;

$$\text{Temperature gradient} = \left[\frac{\theta_2-\theta_1}{L}\right]$$

FACTORS THAT DETERMINE THE RATE OF HEAT FLOW

(i) Temperature gradient.

(ii) Cross-section area (A).

(iii) Nature of the material used (K).

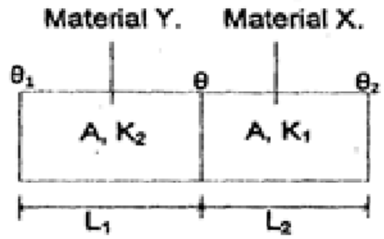
Rate of heat flow is given by

$$\frac{dQ}{dt} = KA\left[\frac{\theta_2-\theta_1}{L}\right]$$

$$K = \frac{\left(\frac{dQ}{dt}\right)}{A\left[\frac{\theta_2-\theta_1}{L}\right]}$$

Where L is the thickness of the material, $(\theta_2 - \theta_1)$ is the temperature difference and K is the coefficient of thermal conductivity of a material.

Rate of heat flow in a composite slab



Consider a perfectly lagged bar made of two materials both of the same cross-sectional area A; the rate of heat flowing through one material is equal to the rate of heat flowing through the other. Therefore, if θ_1 and θ_2 are the temperatures at the ends of materials X and Y respectively and θ is the temperature at the interface, it implies that if $\theta_2 > \theta_1$, then, the rate of heat flow in material X is

$$\frac{dQ}{dt} = k_1 A\left[\frac{\theta-\theta_1}{L_1}\right]$$

$$(\theta - \theta_1) = \left(\frac{L_1}{k_1 A}\right)\left(\frac{dQ}{dt}\right) \quad \ldots\ldots\ldots..(i)$$

Similarly, the rate of heat flow in material Y is

$$\frac{dQ}{dt} = K_2 A\left[\frac{\theta_2-\theta}{L_2}\right]$$

$$(\theta_2 - \theta) = \left(\frac{L_2}{K_2 A}\right)\left(\frac{dQ}{dt}\right) \ldots\ldots\ldots\ldots(ii)$$

Adding the two equations gives

$$(\theta - \theta_1) + (\theta_2 - \theta) = \left(\frac{L_1}{K_1 A}\right)\left(\frac{dQ}{dt}\right) + \left(\frac{L_2}{K_2 A}\right)\left(\frac{dQ}{dt}\right)$$

$$\theta_2 - \theta_1 = \left(\frac{dQ}{dt}\right)\left[\left(\frac{L_1}{K_1 A}\right) + \left(\frac{L_2}{K_2 A}\right)\right]$$

$$\frac{dQ}{dt} = \frac{(\theta_2-\theta_1)}{\left[\left(\frac{L_1}{K_1 A}\right)+\left(\frac{L_2}{K_2 A}\right)\right]} \quad \ldots.(iii)$$

Therefore the rate of heat flow can be calculated from equation *(iii)*. After, the temperature at the interface can then be calculated from either equation (i) or (ii).

NOTE:

The expression ($\frac{L}{KA}$) is known as thermal resistance *R* of a material. For *a* composite slab made of two materials, the total thermal resistance

$R_{total} = \left[\left(\frac{L_1}{k_1 A}\right) + \left(\frac{L_2}{K_2 A}\right)\right]$

Therefore, for a composite slab made of three materials, the total resistance

$$R_{total} = \left[\left(\frac{L_1}{K_1 A}\right) + \left(\frac{L_2}{K_2 A}\right) + \left(\frac{L_3}{K_3 A}\right)\right]$$

Thus, the rate of heat flow can be calculated from

$$\frac{dQ}{dt} = \frac{(\theta_2 - \theta_1)}{\left[\left(\frac{L_1}{K_1 A}\right) + \left(\frac{L_2}{K_2 A}\right) + \left(\frac{L_3}{K_3 A}\right)\right]}$$

Where $\boldsymbol{\theta_2}$ ***and*** $\boldsymbol{\theta_1}$ are the temperatures at the ends of the composite.

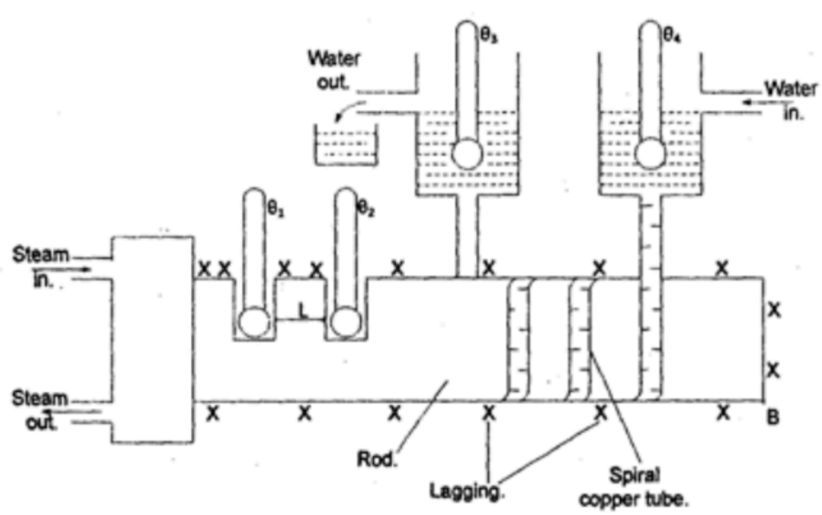
COEFFIIENT OF THERMAL CONDUCTIVITY

$$From\ \frac{dQ}{dt} = KA\left[\frac{\theta_2 - \theta_1}{L}\right]$$

$$k = \frac{\frac{dQ}{dt}}{A\left[\frac{\theta_2 - \theta_1}{L}\right]}$$

Where K is the thermal conductivity of the given material. Thus, thermal conductivity is defined as the rate of heat of heat flow through a material per unit cross-sectional area per unit temperature gradient. S.I unit is W$m^{-1}K^{-1}$.

SEARLE'S METHOD TO DETERMINE THE COEFFICIENT OF THERMAL CONDUCTIVITY OF A METAL ROD (GOOD CONDUCTOR)



The length *l* and diameter *d* of the rod is measured. Its cross-sectional area $A = \left(\frac{\pi d^2}{4}\right)$ is determined. A copper tube carrying running water is wound around end B of the rod. The bar is well lagged and the four thermometers put in place as indicated set up above.

Steam is passed through the steam chest and left to run until steady state conditions are achieved where the temperatures θ_1 , θ_2, θ_3 and θ_4 are steady and all the heat supplied by the steam is being taken by running water.

The water flowing in a spiral copper tube is collected in a time *t.* The temperatures θ_1, θ_2, θ_3 and θ_4 are noted. Let *m* be the mass of water collected in time *t* and *c* be the specific heat capacity of water. Therefore,

$$\begin{pmatrix} rate\ of\ heat\ flow \\ in\ the\ metal\ rod \end{pmatrix} = \begin{pmatrix} rate\ of\ heat\ gained \\ by\ the\ water \end{pmatrix}$$

$$KA\left[\frac{\theta_1-\theta_2}{L}\right] = \frac{mc(\theta_3-\theta_4)}{t}$$

$$K = \left(\frac{mcl}{At}\right)\left(\frac{\theta_3-\theta_4}{\theta_1-\theta_2}\right)$$

Where K coefficient of thermal conductivity of the metal rod Why this method is suitable for good conductors

Why this method is suitable for good conductors

- It requires a measurable rate of heat flow which can be achieved from a good conductor,
- It also requires a steep temperature which can only be got if it's a good conductor.

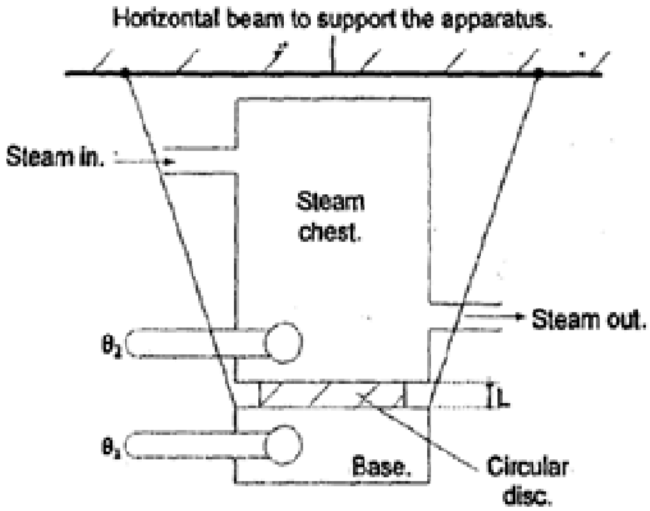
Precautions taken when measuring thermal conductivity of a metal

- The material must have a uniform cross-sectional area.
- The metal rod should be properly/completely lagged,
- The length of the metal rod must be made long compared with its diameter so that a measurable temperature gradient is obtained,
- The two holes for the thermometers must be filled with mercury or oil to ensure thermal contact between the thermometers and the metal rod.
- The surface of the metal rod must be polished to minimize radiation losses,
- The rate of heating should be constant.

LEE'S DISC METHOD TO DETERMINE THE COEFFICIENT OF THERMAL CONDUCTIVITY OF POOR CONDUCTOR

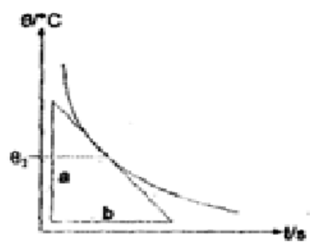
The disc of thickness *L* is measured and the cross-sectional area A is also determined. The mass *m* and the specific heat capacity *c of* the base must be known or measured. The steam is sandwiched between the steam chest and the base. The apparatus is then tied on the string and suspended near an open window. Steam is passed into the steam chest until the temperatures θ_1 and θ_2 are steady. Then the rate at which heat is conducted to the disc is equal to the rate at which heat is lost from the base to the surrounding. That is,

$$\frac{dQ}{dt} = KA\left[\frac{\theta_2 - \theta_1}{L}\right]$$



To determine the rate of heat lost, the disc is removed and the steam heats the base directly till the temperature is 10°C above θ_1. The steam chest is then removed and the disc is put back onto the base. The stop clock is started till the temperature is 5°C below θ_1 recording the time *t* every half a minute.

A graph of temperature against time is plotted and its slope S at a temperature θ_1 is obtained. This slope equals the rate of temperature fall of the base at θ_1.



Slope, $S = \left(\frac{a}{b}\right)$

Rate of heat lost from the metal base at θ_1 = mcS

Thus, at steady state,

$$\left(\begin{matrix}\text{rate of heat conducted} \\ \text{to the metal base}\end{matrix}\right) = \left(\begin{matrix}\text{rate of heat lost} \\ \text{by the base}\end{matrix}\right)$$

$$KA\left[\frac{\theta_2 - \theta_1}{L}\right] = mcS$$

$$K = \left[\frac{mcSL}{A(\theta_2 - \theta_1)}\right]$$

Where *K* is the thermal conductivity of the disc (specimen).

Why this method is suitable for poor conductors

- The large value of surface area and small thickness leads to a measurable rate of heat flow through the specimen (disc),
- The lines of heat flow along the specimen are parallel, that is almost no heat is lost to the surrounding.

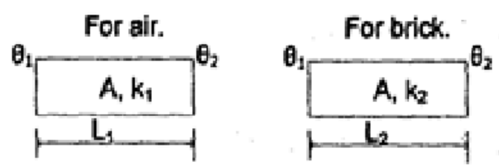
Precautions in the experiment

- The specimen (disc) should be thin in order to obtain a measurable temperature gradient,
- The specimen should have a large cross-sectional area and a small thickness to give adequate rate of heat flow.
- The specimen should be of a disc shape in order to reduce the rate of heat lost to the surrounding.

WORKED EXAMPLES

1. Assuming that thermal conductivity of air and brick is $0.02Wm^{-1}K^{-1}$ and $0.6Wm^{-1}K^{-1}$ respectively; calculate the thickness of air equivalent to a thickness of 30cm of brick if the rate of heat flow in air and brick are the same. If two such brick walls are separated by an air gap of 3cm, how much heat per minute would flow through them in the steady state when the outside temperatures of brick are 60°C and 10°C respectively and the area of cross section of each is $2m^2$.

Solution



$k_1 = 0.02Wm^{-1}K^{-1}, k_2 = 0.6Wm^{-1}K^{-1}, L_2 = 0.3m$

Rate of heat flow through air; $\frac{dQ}{dt} = k_1 A\left[\frac{\theta_1-\theta_2}{L_1}\right]$

Rate of heat flowing through brick; $\frac{dQ}{dt} = K_2 A\left[\frac{\theta_1-\theta_2}{L_2}\right]$

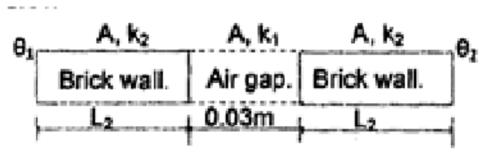
If the rate of heat flow in air and brick are the same;

$$K_1 A\left[\frac{\theta_1-\theta_2}{L_1}\right]=k_2 A\left[\frac{\theta_1-\theta_2}{L_2}\right]$$

$$\frac{K_1}{L_1} = \frac{K_2}{L_2}$$

$$L_1 = \frac{K_1 L_2}{K_2} = \frac{0.02\times0.3}{0.6} = 0.01m$$

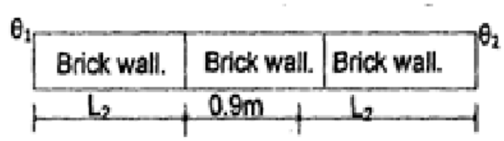
If two such brick walls are separated by an air gap, the illustration is as shown.



$$\theta_1 = 60^oC, \theta_2 = 10^oC, A = 2m^2$$

But 0.01m of air is thermally equivalent to 0.3m of brick.

Therefore, 0.03m of air will be thermally equivalent to $\frac{0.3}{0.01} \times 0.03 = 0.9m$ of brick. The equivalent diagram becomes



$$L = 0.3 + 0.9 + 0.3 = 1.5m$$

Rate of heat flowing through the system

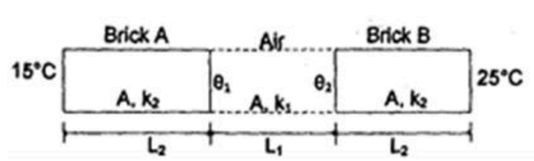
$$\frac{dQ}{dt} = k_2A\left[\frac{\theta_1 - \theta_2}{L}\right]$$

$$\frac{dQ}{dt} = 0.6 \times 2\left[\frac{60-10}{1.5}\right] = 40Js^{-1}$$

$$\frac{dQ}{dt} = 40Js^{-1} = 40 \times 60 = 2400\ J/min$$

2. A cavity wall is made of two brick layers each of thickness 10cm and between which there is a layer of air 2cm thick. If the thermal conductivity of air is 0.024W$m^{-1}K^{-1}$and that of brick is 0.7W$m^{-1}K^{-1}$and that the inner and outer walls have temperatures 25^{0}C and 15^{0}C respectively, find the rate of heat flow through 1m^2 of the wail.

Solution



$$L_1 = 0.02m, K_1 = 0.024Wm^{-1}K^{-1}, K_2 = 0.7Wm^{-1}K^{-1}, L_2 = 0.1m, A = 1m^2$$

For brick A, the rate of heat flow

$$\frac{dQ}{dt} = k_2A\left[\frac{\theta_1 - 15}{L_1}\right]$$

$$(\theta_1 - 15) = \left(\frac{L_1}{K_1A}\right)\left(\frac{dQ}{dt}\right) \quad ....(i)$$

Similarly, for air; $(\theta_2 - \theta_1) = \left(\frac{L_2}{K_2A}\right)\left(\frac{dQ}{dt}\right)$ ... (ii)

Also, for brick B; $(25 - \theta_2) = \left(\frac{L_2}{K_2A}\right)\left(\frac{dQ}{dt}\right)$ .. (iii)

Adding equation (i), (ii) and (iii) gives

$$(\theta_1 - 15) + (\theta_2 - \theta_1) + (25 - \theta_2) = \left(\frac{L_1}{K_1A}\right)\left(\frac{dQ}{dt}\right) + \left(\frac{L_2}{K_2A}\right)\left(\frac{dQ}{dt}\right) + \left(\frac{L_2}{K_2A}\right)\left(\frac{dQ}{dt}\right)$$

$$(25 - 15) = \left(\frac{dQ}{dt}\right)\left[\left(\frac{L_1}{K_1A}\right) + \left(\frac{L_2}{K_2A}\right) + \left(\frac{L_2}{K_2A}\right)\right]$$

$$10 = \left(\frac{dQ}{dt}\right)\left[\left(\frac{0.02}{0.024}\right) + \left(\frac{0.1}{0.7\times1}\right) + \left(\frac{0.1}{0.7\times1}\right)\right]$$

$$10 = \left(\frac{dQ}{dt}\right)\left[\frac{47}{42}\right]$$

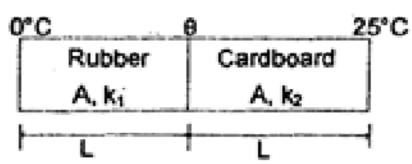
$$\left(\frac{dQ}{dt}\right) = \frac{42}{47} \times 10 = 8.936\,Js^{-1}$$

3. A composite, heat of rubber and cardboard each of 2mm thick are placed in contact and their outer faces maintained at 0°C arid 25°C respectively. If the thermal conductivity of rubber is $0.13Wm^{-1}K^{-1}$ and that of cardboard is $0.05Wm^{-1}K^{-1}$ and area of the rod is $100cm^2$. Calculate:

(i) The temperature of the junction.

(ii) The rate of heat flow in one hour.

Solution



$$L = 0.002m, A = 0.01m^2, k_1 = 0.13Wm^{-1}K^{-1}, k_2 = 0.05Wm^{-1}K^{-1}$$

$$for\ rubber, \frac{dQ}{dt} = K_1A\left[\frac{\theta - 0}{L}\right] \quad ...(i)$$

$$for\ the\ cardboard, \frac{dQ}{dt} = K_2A\left[\frac{25-\theta}{L}\right] \quad ...(ii)$$

Equating (i)and(ii)gives;

$$k_1A\left[\frac{\theta - 0}{L}\right] = K_2A\left[\frac{25 - \theta}{L}\right]$$

$$k_1\theta = k_2(25 - \theta)$$

$$0.13\theta = 0.05(25 - \theta)$$

$$0.18\theta = 1.25$$

$$\theta = 6.94^oC$$

(ii) From (i)

$$\frac{dQ}{dt} = k_1A\left[\frac{\theta-0}{L}\right] = 013 \times 0.01 \times \frac{6.94}{0.002} = 4.511Js^{-1}$$

$$\frac{dQ}{dt} = 4.511 \times 3600 = 16239.6\,joules\ per\ hour$$

$$\frac{dQ}{dt} = 16239.6\,joules\ per\ hour$$

Trial Questions

1. A concrete floor of a hall has dimensions of 10m by 8m. It's covered with a carpet of thickness 2cm. The temperature inside the hall is 22°C while that of the surroundings just below the concrete is 12°C. The thermal conductivities of the concrete and the material of the carpet are $1Wm^{-1}K^{-1}$ and $0.05Wm^{-1}K^{-1}$ respectively and the thickness of the concrete is 10cm. Calculate,

(i) The temperature of the interface of the concrete and the carpet.

(ii) The rate at which heat flows through the floor. [Ans: 14.04^0C , 1600W]

2. A cooking utensil of thickness 3mm is to be made from two layers; one of aluminum and the other of brass. If one layer is to be 2mm and the other 1mm, determine which combination allows higher rate of flow of heat.[Coefficient of thermal conductivity for brass is $112Wm^{-1}K^{-1}$ while that for aluminum is $240Wm^{-1}K^{-1}$]

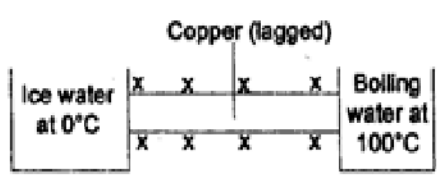
[Ans: The thickness should be 2mm for aluminium and 1mm for brass]

3. A closed metal vessel contains water;
(i)At 30°C (ii) At 75°C

A vessel has surface area of $0.5m^2$ and uniform thickness of 4mm. If the outside temperature is 15°C, calculate the heat loss per minute of the vessel trough conduction.

[Thermal conductivity is 400 $Wm^{-1}K^{-1}$] [Ans:$4.5 \times 10^7 Jmin^{-1}$, $1.8 \times 10^8 Jmin^{-1}$]

4. A well lagged copper bar between boiling water and ice-water mixture is 12cm long.



(i) Calculate the rate of flow of heat through the copper bar.

(ii) Calculate the mass of ice which will melt during 15s. [K for copper = 385 $Wm^{-1}K^{-1}$, A for the bar = $1.5cm^2$, Latent heat of fusion of ice = $3.34 \times 10^5 kg^{-1}$]

[Ans: 48.125W, 2.16g]

5. A copper kettle has a base of thickness 2mm and area $3 \times 10^{-2}m^2$. Estimate the steady difference in temperature between the inner and outer surface of the base which must be maintained to enable enough heat to pass through so that the temperature of 100kg of water rises at a rate of $0.25Ks^{-1}$ .Take c for water = $4200Jkg^{-1}K^{-1}$,for copper = 380 $Wm^{-1}K^{-1}$]

[Ans: 18.42K]

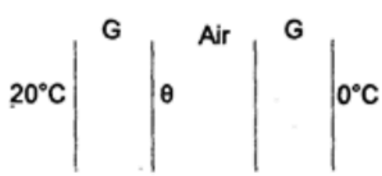
6. A wall 6m by 3m consists of two layers of bricks of thermal conductivities 0.6 $Wm^{-1}K^{-1}$ and 0.5 $Wm^{-1}K^{-1}$ respectively. The thickness of each layer is 15cm. The inner surface layer A is at 20°C while the outer layer B is at a temperature of 10°C. Calculate the temperature of interface of A and B; and the rate of flow of heat through a wall.

[Ans: 15.45^0C , 327.6W]

7. A window of height 1m and width 1.5m contains a double glazed unit consisting of two single glass panes each of distance 2mm separated by air. Calculate the rate at which heat is conducted through the window if the temperatures of the external surfaces of glass are 20°C and 30°C.[K for glass = 0.72 $Wm^{-1}K^{-1}$, K for air =0.025 $Wm^{-1}K^{-1}$, Thickness of each glass = 4mm]
[Ans: 164.63W]

8. A cooking saucepan made of iron has a base area of $0.05m^2$ and thickness of 2.5mm. It has a thin layer of soot of average thickness 0.5mm on its bottom surface. Water in the saucepan is heated until it boils at 100°C. The water boils away at a rate of 0.6kg per minute and the side of the soot nearest to the heat source is at 150°C. Find the thermal conductivity of soot. [K for iron = 66 $Wm^{-1}K^{-1}$, Specific latent heat of vaporization = $2200kJkg^{-1}$] [Ans: 6.6 $Wm^{-1}K^{-1}$]

9. In double glazing, two thickness of glass G, each 20mm thick are separated by 100mm thickness of air as shown below



The thermal conductivities of glass and air are respectively 1,0 $Wm^{-1}K^{-1}$ and 2.0 $Wm^{-1}K^{-1}$ and the outer surfaces of the glass are 20°C and 0°C respectively .Calculate:
(i) The thickness of the glass which is thermally equivalent to 100m thickness of air.
(ii) The rate of heat flow per unit area through the glass and air. [Ans: 5m, 3.97Wm^{-2}]

10. In order to minimize heat losses from a glass container, the walls of the container are made of two sheets of glass each 2mm thick, placed 3mm apart; the intervening space being filled with a poorly conducting solid. Calculate the ratio of the rate of conduction of heat per unit area through this composite wall to that which would have occurred had a single sheet of the same

glass been used under the same internal and external conditions of temperature. [Assume the conductivity of glass and poorly conducting solid is 0.63 $Wm^{-1}K^{-1}$ and 0.049 $Wm^{-1}K^{-1}$ respectively] [Ans: 7 : 149]

11. A cavity wall is made of bricks 0.1m thick with an air space 0.1m thick between them. Assuming the thermal conductivity of brick is 20 times that of air, calculate
(i) The thickness of brick which conducts the same quantity of heat per second per unit area as 0.1m of air.
(ii) If the thermal conductivity of brick is 0.5 $Wm^{-1}K^{-1}$, calculate the rate of heat conducted per unit area through the cavity wall when the outside surfaces of the brick walls are respectively 19°C and 4°C.

CHAPTER 16: THERMAL RADIATION

Thermal radiation is a form of energy emitted by a body in form of electromagnetic waves because of its temperature. All bodies emit some kind of some kind of radiation but the intensity and wave length of the radiation varies depending on the temperature of the body

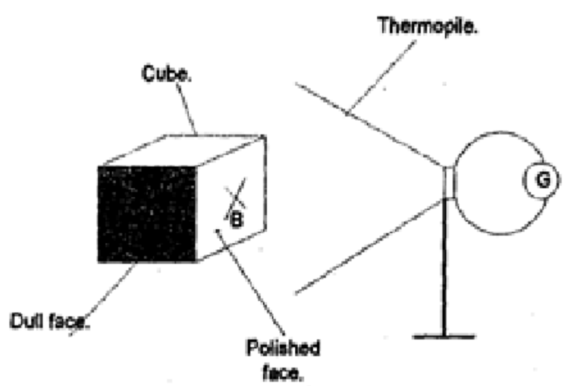
Examples of thermal radiation include: infrared, visible light and ultra-violet radiation. The sun is the main source of radiation energy.

FACTORS ON WHICH THERMAL RADIATION DEPENDS

The amount of radiation emitted by a body depends upon three factors:

- Surface area of the body.
- The temperature of the body.
- The nature of the surface of the body.

Experiment to compare the energy radiated by dull and polished surfaces



A cube, painted with one of the faces having a polished surface and the other having a black surface is filled with hot water and placed near a thermopile. The cube is turned such that the black surface faces the thermopile. A deflection is observed on the galvanometer and noted.

The cube is then turned such that the polished surface faces the thermopile. The deflection is again observed on the galvanometer and noted.

The deflection caused by the black surface will be observed to be greater than that caused by the polished surface. Therefore, dull surfaces radiate more energy than polished surfaces.

NOTE: This experiments show that black surfaces are the best emitters and absorbers of radiation at a given temperature.

ULTRA-VIOLET RADIATION

Sources of ultra-violet radiation

- The sun.
- Current through mercury vapour.

Effects of ultra-violet radiation

- They cause photoelectric emission.
- They cause fluorescence.
- They can cause skin burn.
- They can affect a photographic film.

INFRA-RED RADIATION

Radiation of longer wavelength than the visible light (ranging from 0.7μm to about 1mm) in the electromagnetic spectrum is called infra-red.

Sources of infra-red radiation

- The sun.
- Hot bodies.

Effects of infrared radiation

- They cause photoelectric emission.
- They cause a sensation of warmth or heat on a body.
- Like all other radiations, infra-red can be refracted or reflected.

Applications of infra-red radiations

- Night vision: Helicopters fitted with infra-red detectors can be flown at night with clear view of the ground.
- Radio photography: Clear photographs can be taken at night using infra-red lenses and detectors.

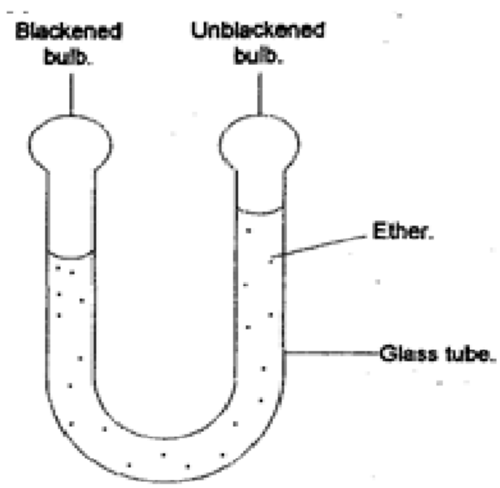
- Remote-control keypads for televisions and radio recorders use infra-red signaling,
- They are used to dry paint on cars during manufacture.

DETECTION OF INFRARED RADIATION

Since infra-red has a longer wavelength than the visible light, the eye can't detect infra-red radiation. Detectors of infra-red radiation therefore include:

- Ether thermoscopes.
- Bolometer.
- Thermopiles.
- Optical pyrometers.

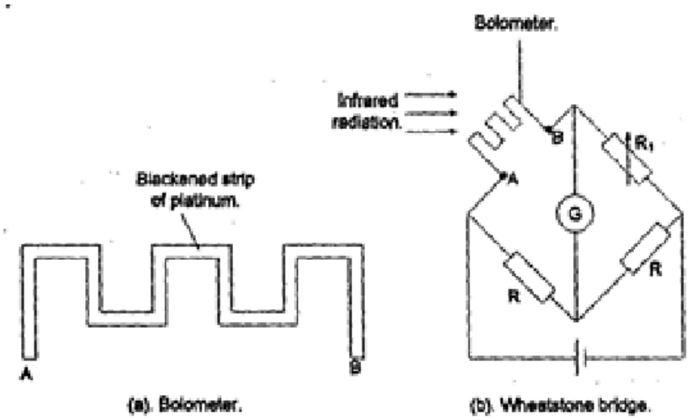
ETHER THERMOSCOPE



The tube is partly filled with ether and therefore both bulbs contain a mixture of air and ether vapour. When infra-red radiation falls on the thermoscope, the blackened bulb, which is a good absorber of radiation, absorbs more of the infra-red radiation than the un-blackened bulb.

This makes the air-vapour mixture in the blackened bulb to increase, causing it to expand and thus increasing the pressure in the blackened bulb. The increased pressure in the blackened bulb pushes the ether along the tube and this therefore shows the presence of infra-red radiation.

BOLOMETER



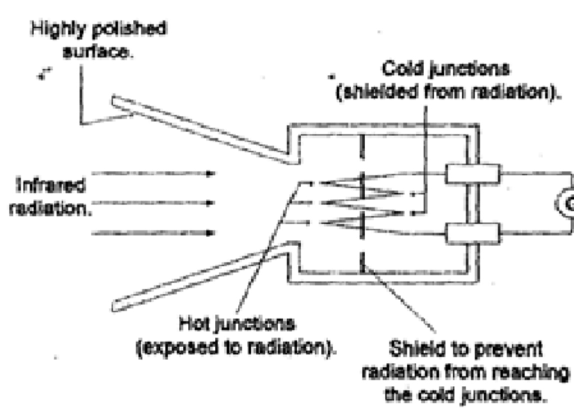
A bolometer consists of a blackened strip of platinum foil arranged in a zigzag pattern as shown in the figure (a) above. The strip is then connected to a Wheatstone bridge as shown in figure (b) above.

When infra-red radiation falls on the bolometer, heat is gained by the platinum strip and this increases the temperature and hence, resistance of the strip.

The increase in resistance can be seen by a deflection on the galvanometer G. This shows the presence of infra-red radiation. Thus, infra-red radiation has been detected.

THERMOPILE

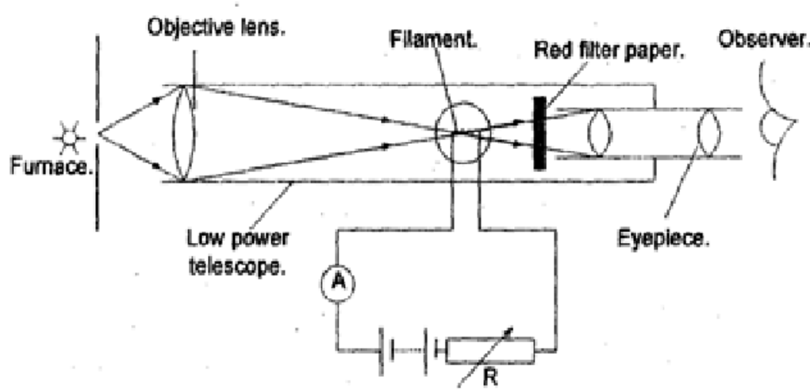
A thermopile consists of a number of thermocouples connected in series with one set of the junctions exposed to the radiation and the other shielded from radiation. The junctions which are exposed to radiation are blackened to make then good absorbers. When infra-red radiation falls on the thermopile, the highly polished metal cone concentrates the radiation onto the exposed junctions of the thermocouples. The exposed junctions become hot while the shielded ones remain cold.



A thermoelectric e.m.f, which depends on the temperature of the junctions, is generated and can be seen by a deflection on the galvanometer G. This shows the presence of infra-red radiation. Thus, infra-red radiation has been detected.

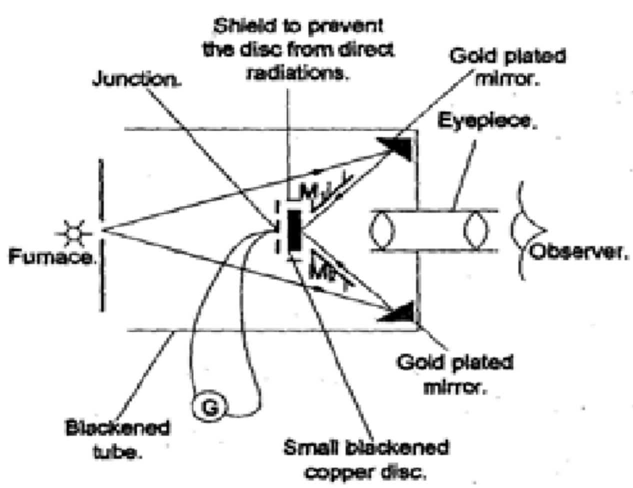
OPTICAL RADIATION PYROMETER

This can be used to detect radiations from very hot bodies. Thus, they can be used to measure furnace temperature. The source is focused using the objective lens such that its image lies in the same plane with that of the filament. The observer looks through the eyepiece to observe the image of the small area of object focused on the filament.



If the image of the hot body is brighter than the filament, the filament appears dark on a bright background; implying that the filament is at a lower temperature than that of the hot body. The temperature of the filament is adjusted using a rheostat till the image of the hot body has the same brightness as the filament. The ammeter is calibrated to read the temperature of the hot body. Thus the temperature of the hot body can be read on the ammeter.

TOTAL RADIATION PYROMETER



The eyepiece is focused on a small blackened disc and the gold plated mirrors adjusted until the radiations from the furnace are focused on the disc. Small plane mirrors M_1 and M_2 make the focusing easier.

The radiations from the source warm the junction attached to the disc and a thermoelectric e.m.f is set up. The galvanometer is calibrated in degrees and so the temperature can be read off.

GREEN HOUSE EFFECT IN RELATION TO GLOBAL WARMING

Radiations from the sun have short wave length due to the high temperature of the sun. When this radiation, of short wave length passes through the water vapour and carbon dioxide gas in the lower layers of the atmosphere, it warms up the lower layers of the earth.

The earth will then re-emit this radiation (infra-red radiation) as a black body radiation of long wavelength because of the low temperature of the earth.

The black body radiation will therefore be trapped by the water vapor and carbon dioxide gas in the atmosphere thus preventing the radiation from escaping from the earth's atmosphere.

The radiation which has been prevented from escaping from the earth's atmosphere will therefore cause global warming.

NOTE: If the amount of carbon dioxide in the atmosphere increases, the average temperature of the earth increases because more radiation, which is re-emitted, will be trapped. This will cause dramatic climatic and geographic change to occur.

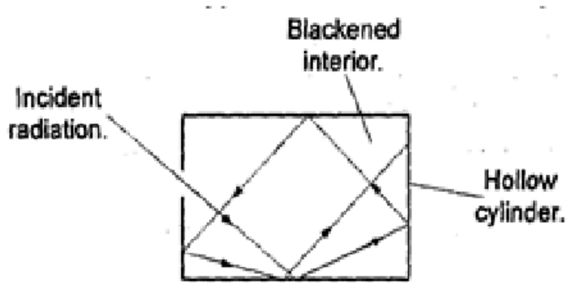
BLACK BODY RADIATION

A black body is a perfect absorber of all the radiation which is incident on it.

A black body radiator is one which emits a radiation which is characteristic of its temperature and in particular which doesn't depend on the nature of its surfaces.

Black body radiation is the radiation from a black body.

How to make an approximation of a black body



A small hole is made in an enclosure e.g. a hollow cylinder whose inner walls are painted black. When radiation enters into the cylinder through the small hole, it undergoes multiple reflections. Since black surfaces are good absorbers, a certain proportion of radiation will be absorbed at each reflection till virtually all the radiation is absorbed by the black inner walls of the cylinder. The hole in the enclosure therefore approximates a black body.

QUALITY OF RADIATION

It refers to the relative intensities of different wavelengths i.e. the proportion of red to blue in the radiation. Relative intensity is defined as the energy radiated per second per unit surface area of the body in a unit wavelength interval. The quality of radiation depends on the temperature of the body. Suppose an iron bar is thrust into a fire. At first, it will look black though it has become hot. This is because the radiation given out by the bar is entirely in the infra-red region. As the temperature increases, iron bar becomes red hot. Now the bar gives out more radiation than before. A larger portion of it is in the visible region than before. As the temperature rises, the bar .appears orange, then yellow and finally white. Thus, the nature of the radiation given out by the bar and the relative amounts of the radiations of different wave lengths depend on the temperature of the radiator.

Qn. Explain why the deepest part (centre) of a fire looks brightest (white)

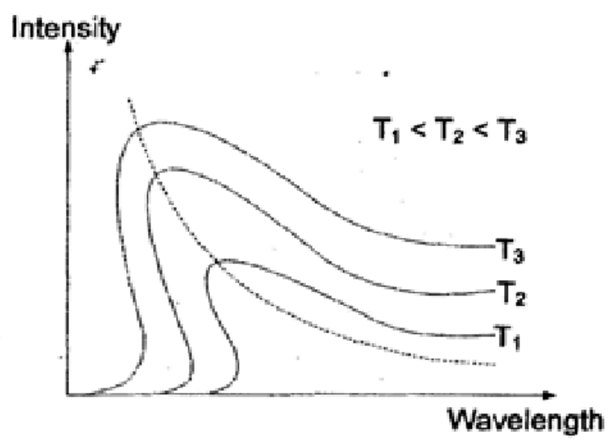
Soln. At the centre of a fire, temperature is highest. Since the quality of radiation (relative intensity of different wavelengths) depends on temperature, the relative intensity is therefore maximum at the deepest part of the fire.

This maximum relative intensity corresponds to all the different wavelengths being emitted and since these wavelengths are in the visible region, a combination of all these colors results into white.

DISTRIBUTION OF ENERGY WITHIN THE SPECTRUM OF A BLACK BODY

(SPHERAL DISTRIBUTION OF BLACK BODY RADIATION)

A graph of relative intensity against wavelength illustrates the distribution of energy in black body radiation.



Features (observations) of the curves shown

- The curves at lower temperatures lie completely inside those at higher temperatures,
- The intensity for each wavelength increases with increase in wavelength but the increase is faster on the side of the shorter wavelength,
- At each temperature *T,* the energy emitted is maximum for a certain wave length λ_{max} which decreases with rising temperature.

NOTE: The dotted line shows the locus of the peaks of the curves for the different temperatures.

Conclusions from the above observations

- A body radiates energy at a rate which is determined by the nature of the surface and its temperature,
- A body absorbs energy at a rate determined by the nature of the surface and the temperature of the surroundings,
- The area under any particular curve is the total energy radiated per unit surface area per second; at the corresponding temperature.

LAWS OF BLACK BODY RADIATION

1. **Wien's displacement law**

It states that the product of maximum wavelength, λ_{max} and the absolute temperature, *T* of a body is equal to a constant. That is,

$$\mathrm{T}\lambda_{max} = constant$$

The constant is called Wien's constant and has a value 2.9 x 10^{-3}mK.

2. **Stefan's law**

It states that the total energy radiation per unit surface area per second (E) of a black body is proportional to the fourth power of the absolute temperature *T* of the body. That is,

$$E \propto T^4$$

$$E = \sigma T^4$$

Where σ is Stefan's constant and has a value 5.67 x $10^{-8} W m^{-2} K^{-4}$.

If the body is of surface area *A,* then the total energy radiated per second is

$$E_{total} = \sigma A T^4$$

But energy radiated per second is equal to power. The above equation can therefore be written as; $P = \sigma A T^4$

PREVOST'S THEORY OF HEAT EXCHANGE

It states that a body radiates or absorbs heat energy at a rate determined by the nature of the surface and the temperature of the surroundings.

Heat exchanges for a body in an enclosure

The rate of loss or gain of heat (radiation) from a body depends on the temperature of its surrounding.

If the body's temperature is higher than that of the surrounding, it loses energy by radiation to the surrounding until dynamic equilibrium is reached. If the body's temperature is lower than that of the surrounding, it gains heat energy until dynamic equilibrium is reached. At dynamic equilibrium, the rate of heat loss is equal to the rate of heat gain.

Consider a black body whose absolute temperature is T in an enclosure at absolute temperature

If $T > T_0$, the black body loses energy by radiation to the surrounding until dynamic equilibrium is reached. At dynamic equilibrium, the net rate of energy lost is given by

$$P_{lost} = \sigma A(T^4 - T_0^4)$$

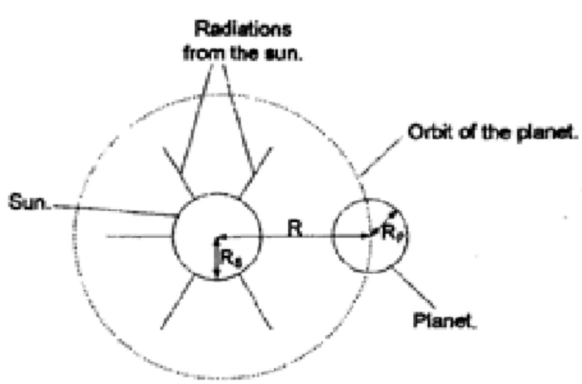
If $T < T_0$, the black body gains heat energy by from the surrounding until dynamic equilibrium is reached. At dynamic equilibrium, the net rate of energy gained is given by

$$P_{gained} = \sigma A(T_0^4 - T^4)$$

Assumption made is that the body is a black and that the interchange of heat is by radiation.

Emission and absorption of heat energy by planets in the universe

The sun is the centre of the universe. Planets revolve round the sun in elliptical orbits. The sun and the planets are assumed to be perfect spheres. Also, the sun is assumed to radiate heats as a black body. The planets are also assumed absorb and emit radiations as black bodies. The figure shows a sketch of one of the planets revolving round the sun in the universe.



Where R_s is the radius of the sun, R_p is the radius of the planet, and R is the mean distance between the sun and the planet. Therefore,

Surface area of the sun is given by, $A_s = 4\pi {R_s}^2$

Surface area of the planet is given by, $A_p = 4\pi {R_p}^2$

Total area on which the sun's radiation falls at a distance R from the sun where the planet is situated is given by, $A = 4\pi R^2$.

Effective area of the planet on which the sun's radiation is incident normally is given by;

$$A_{ef} = \pi {R_p}^2$$

If T_s is the temperature at the surface of the sun implies that the total power radiated by the sun is given by, $P_s = \sigma A_s {T_s}^4$

The power received by the total area on which the sun's radiation falls is equal to the power radiated by the sun. Therefore,

An area A (in square meters) receives P_s watts from the sun. Therefore an area of A_{ef} (in square meters) receives $\left[\frac{P_s}{A} \times A_{ef}\right]$ Watts from the sun.

Thus, power received by the planet is given by

$$P_{recieved} = \left[\frac{P_s}{A} \times A_{ef}\right] = \left[\left(\frac{\sigma A_s {T_s}^4}{4\pi R^2}\right) \times A_{ef}\right]$$

$$= \left[\left(\frac{\sigma (4\pi R_s^2){T_s}^4}{4\pi R^2}\right) \times A_{ef}\right]$$

$$P_{recieved} = \left[\sigma \left(\frac{R_s}{R}\right)^2 T_s^4 \times A_{ef}\right]$$

Therefore,

$$\begin{pmatrix} power\ radiated \\ by\ the\ planet \end{pmatrix} = \left[\sigma \left(\frac{R_s}{R}\right)^2 T_s^4 \times A_{ef}\right], where\ A_{ef} = \pi R_p^2$$

Assuming radiative equilibrium,

$$\begin{pmatrix} power\ radiated \\ by\ the\ planet \end{pmatrix} = \begin{pmatrix} power\ recieved \\ by\ the\ planet \end{pmatrix}$$

$$P_{radiated} = P_{received}$$

$$\sigma A_p T_p^4 = \left[\sigma \left(\frac{R_s}{R}\right)^2 T_s^4 \times A_{ef}\right]$$

$$\sigma\left(4\pi R_p^2\right)T_p^4 = \left[\sigma\left(\frac{R_s}{R}\right)^2 T_s^4 \times \left(\pi R_p^2\right)\right]$$

$$T_p^4 = \left(\frac{R_s}{2R}\right)^2 T_s^4$$

$$T_p = T_s \times \sqrt{\left(\frac{R_s}{2R}\right)}$$

Where T_p *is the temperature at the surface of the planet.*

NOTE:
When the sun radiates heat, its total mass decreases and the decrease in mass is known as mass defect (m). The expression relating mass defect and the total power radiated by the sun is given by;

$$P_s = (\Delta m)c^2$$

Where c is the velocity of light in a vacuum and has a value 3 x $10^8 ms^{-1}$.

Therefore, the total loss of mass by the sun is given by;

$$\Delta m = \left[\frac{P_S}{C^2}\right]$$

WORKED EXAMPLES

1. Calculate the net rate of loss of heat energy from a spacecraft of surface area $25m^2$ and at a temperature of 300K, if the radiation it receives from the sun is equivalent to the temperature in space of 50K. Assume the space craft is a black body.

Solution

A = $25m^2$, T = 300K, T_o = 50K

$$P_{lost} = \sigma A\left(T^4 - T_0{}^4\right)$$

$$P_{lost} = 5.67 \times 10^{-8} \times 25(300^4 - 50^4) = 11{,}472.89W$$

2. The total power output of the sun is 4 x 10^{26}W. Given that the mass of the sun is 1.97 x 10^{31} *kg* and its density is 1.4 x $10^3 kgm^{-3}$. Estimate the temperature of the sun. State any assumptions made.

Solution

Assumption: it's assumed that the sun is a perfect black body and that it's spherical in shape.

$$(volume\ of\ the\ sun) = \left[\frac{mass}{density}\right]$$

$$\left(\frac{4}{3}\pi R_S^3\right) = \left[\frac{m}{\rho}\right]$$

$$\frac{4}{3}\pi R_S^3 = \frac{1.97\times10^{31}}{1.4\times10^{3}}$$

$$R_S^3 = \frac{3\times1.97\times10^{31}}{4\pi\times1.4\times10^{3}} = 3.36\times10^{27}$$

$$R_S = \sqrt[3]{3.36\times10^{27}} = 1.498\times10^{9}m$$

Surface area of the sun

$$A_S = 4\pi R_S^2 = 4\pi(1.498\times10^{9})^2 = 2.82\times10^{19}m^2$$

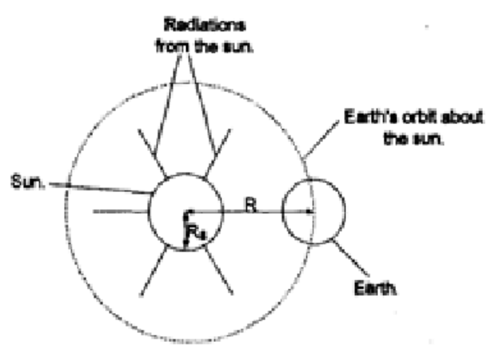
Total power output of the sun, $P_s = \sigma A_s T_s^4$

$$T_S^4 = \frac{P_S}{\sigma A_S} = \left[\frac{4\times10^{26}}{5.67\times10^{-8}\times2.82\times10^{19}}\right] = 2.502\times10^{14}$$

$$T_s = \sqrt[4]{2.502\times10^{14}} = 3977\text{K}$$

3. The energy intensity received by the earth from the sun is about 1400Wm^{-2}. Estimate the surface temperature of the sun.

Solution



$$R_s = 7\times10^{8}m, R = 1.5\times10^{11}m$$

$$\binom{power\ recieved}{by\ the\ earth} = \binom{energy}{intensity} x \binom{area\ of\ orbit}{containing\ the\ earth}$$

$$P_{recieved} = 1400\times4\pi\times(1.5x10^{11})^2 = 3.958\times10^{26}W$$

At equilibrium,

$$\binom{power\ emitted}{by\ the\ sun} = \binom{power\ recieved}{by\ the\ earth}$$

$$P_S = P_{recieved} = 3.958\text{ X }10^{26}W$$

Surface area of the sun;

$$A_s = 4\pi R_s^2 = 4\pi(7 \times 10^8)^2 = 6.158 \times 10^{18} m^2$$

Total power emitted by the sun

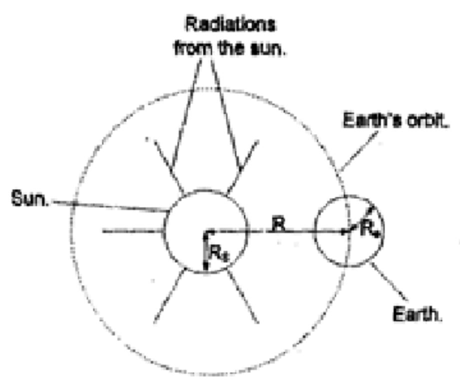
$$P_s = \sigma A_s T_s^4$$

$$T_S^4 = \frac{P_S}{\sigma A_S} = \left[\frac{3.958 \times 10^{26}}{5.67 \times 10^{-8} \times 6.158 \times 10^{18}}\right] = 1.134 \times 10^{15}$$

$$T_S = \sqrt[4]{1.134 \times 10^{15}} = 5802K$$

4. The sun is a black body of surface temperature 6000K.
(i) Calculate the amount of radiant energy approaching the earth from the sun.
(ii) Why would the actual energy be less than that obtained in (i) above?
(iii) Calculate the earth's temperature.

Solution



(i). Surface area of the sun is given by

$$A_s = 4\pi {R_s}^2 = 4\pi(7 \times 10^8)^2 = 6.158 \times 10^{18} m^2$$

Surface area of the earth is given by

$$A_e = 4\pi {R_e}^2 = 4\pi(6.4 \times 10^6)^2 = 5.147 \times 10^{14} m^2$$

Total surface area of the earth's orbit about the sun

$$A = 4\pi R^2 = 4\pi(1.5 \times 10^{11})^2 = 2.827 \times 10^{23} m^2$$

Effective area of the earth on which the sun's radiation is incident normally is given by

$$A_{ef} = \pi R_e^2 = \pi(6.4 \times 10^6)^2 = 1.287 \times 10^{14} m^2$$

The total power radiated by the sun is given by;

$$P_s = \sigma A_s T_s^4 = 5.67 \times 10^{-8} \times 6.158 \times 10^{18} \times 6000^4$$

$$= 4.525 \times 10^{26} W$$

Thus, power received by the earth is given by;

$$P_{recieved} = \left[\frac{P_s}{A} \times A_{ef}\right] = \frac{4.525 \times 10^{26}}{2.827 \times 10^{23}} \times 1.287 \times 10^{14}$$

$$= 2.06 \times 10^{17} W$$

(ii) This is because solar radiation incident on the atmosphere is partly absorbed by atmospheric gases and partly scattered. This accounts for the difference.

(iii)

$$T_e^2 = \frac{R_s}{2R} \times T_s^2 = \frac{7 \times 10^8}{2 \times 1.5 \times 10^{11}} \times (6000)^2 = 8.4 \times 10^4$$

$$T_e = \sqrt{8.4 \times 10^4} = 289.8K$$

Trial Questions

1. The filament of a bulb, 0.5m long has a radius of 10mm. the filament melts when it's connected across 240V and a current through it is 0.4A. Calculate:

(i) The temperature at which the filament melts,

(ii) The wavelength of which the energy radiated per second is maximum.

[Ans: 481.82K, $6.02 \times 10^{-6} m$]

2. Calculate the rate of loss of heat energy of a blackbody of area $40m^2$ at a temperature of 70°C; if the radiation it receives from the sun is equivalent to a temperature in space of -200°C.

[Ans: $3.133 \times 10^4 W$]

3. A metal sphere with a blackbody surface and radius 30mm is cooled to -73°C and' is placed inside an enclosure at a temperature of 27°C. Assuming that the density of metal is $800kgm^{-3}$ and that its specific heat capacity is $400kJkg^{-1}K^{-1}$, calculate the initial rate of temperature rise of the sphere. [Ans: $0.115Ks^{-1}$]

4. A black hemisphere of radius 2cm is contained in a hollow evacuated enclosure which is maintained at 27°C. Assuming that the sphere radiates likes a blackbody, calculate the rate at which the sphere loses heat when it's 227°C. [Ans: 15.5W]

5. Estimate the temperature of the earth assuming it's in radiative equilibrium with the sun. Take the temperature of the solar surface to be 6000K. [Ans: 289.82K]

6. The normal flux of radiant energy from the sun at the earth surface is $1360Wm^{-2}$. Calculate:

(i) The total power emitted by the sun.

(ii) The temperature of the sun.

(iii) The rate of loss of mass by the sun.

[Ans: $3.845 \times 10^{26} W$, $5760.72K$, $4.27 \times 10^{9} kgs^{-1}$]

7. A roof of dimensions 20m by 50m is blackened. If the temperature of the sun's surface is 6000K, calculate the solar energy incident on the roof per minute assuming that half of the energy is lost in passing through the earth's atmosphere and that the roof is normal to the sun's rays. [Ans: $5.9 \times 10^{7} J\ per\ minute$]

8. Calculate the rate of loss of heat energy of a blackbody of area 40m at a temperature of 50°C. If the radiation it receives from the sun is equivalent to a temperature in space of -220°C. [Ans: 616.706W]

9. A solid copper sphere of diameter 10mm and temperature 150K is placed in an enclosure maintained at a temperature of 290K. Assuming that the density of copper is $8.93 \times 10^{3}\ kgm^{-3}$ and that its specific heat capacity is $3.7 \times 10^{2}\ Jkg^{-1}K^{-1}$, calculate the initial rate of temperature rise of the sphere. [Ans: $0.069Ks^{-1}$]

10. The solar radiation falling normally on the surface of the earth has an intensity of $1.40kWm^{-2}$. If this radiation fell normally on one side of a thin freely suspended blackened metal plate and the temperature of the surrounding was 300K, calculate the equilibrium temperature of the plate. Assume all interchange is by radiation.

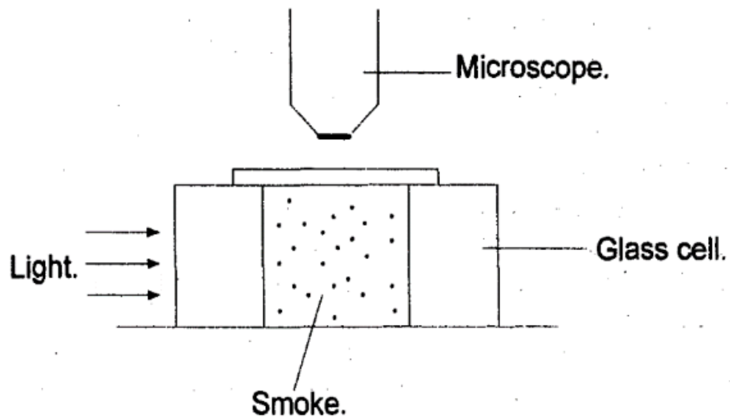
[Ans: 378.138K]

CHAPTER 17: KINETIC THEORY AND IDEAL GAS

KINETIC THEORY OF MATTER

It states that matter consists of tiny particles called molecules which are always in constant vibration in solids, continuous and random motion in liquids and gases; and possess energy. If supplied, heat increases the energy of the molecules or particles hence their random velocity increases.

Experiment to support kinetic theory of matter



Smoke is confined in a glass cell and light is made incident on it as shown above. When viewed through a high power microscope, smoke is seen to be in continuous random motion and this motion goes on requiring no supply of energy from outside.

Something must be knocking the smoke particles and this something is the molecules of the supporting fluid/matter (which is air in this case). Thus, tiny particles of matter are in continuous random motion.

When temperature is increased, the motion of smoke particles is more rapid. At the same temperature, the motion is more rapid for smaller particles than for larger ones.

How kinetic theory of matter accounts for the three states of matter

Kinetic theory of matter consists of three states of matter as having tiny particles called molecules.

In solids, molecules are in constant vibration about their mean position, held together and tightly by stronger intermolecular forces of attraction. Molecules are therefore arranged very close together in regular patterns thus making solids to have definite shape.

In liquids, the intermolecular forces of attraction aren't strong enough to hold molecules in fixed positions. Molecules are therefore in continuous random motion and this freedom of movement enables liquids to take up the shape of any vessel in which they are placed. However, though the intermolecular forces are weaker, they are able to prevent the liquid from going too far apart thus making the liquids to have a definite volume but no definite shape.

In gases, the intermolecular forces are extremely weak making the molecules to be much farther apart than in solids and liquids. They move everywhere in the container containing the gas with a kinetic energy proportional to the absolute temperature and they fill the gas they are in. hence gases have a definite shape and size.

KINETIC THEORY OF GASES

Gases are made up of small particles called molecules which are held by very weak intermolecular forces of attraction. The molecules are in continuous random motion and continuously collide with themselves and the walls of the container; making perfect collisions.

The higher the temperature, the faster the molecules are moving and the greater their kinetic energy or internal energy is.

Using kinetic theory of gases to explain absolute zero temperature

Absolute zero is defined as the temperature at which the molecules have their lowest possible kinetic energy.

According to kinetic theory of gases, molecules of a gas have an average speed which increases with temperature. So, as a gas is cooled, its molecules move more and more slowly and hence their kinetic energy decreases more and more. A point is reached when the molecules are assumed to come to rest and their kinetic energy becomes zero. At this point the gas has the lowest possible temperature called absolute zero.

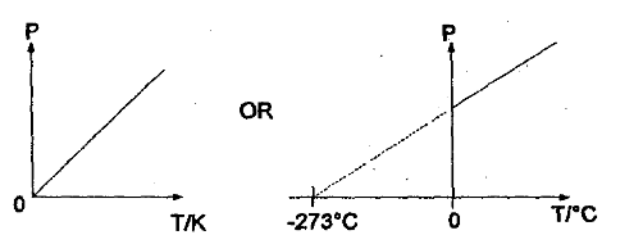
GAS LAWS

The gas laws include:

- Pressure law.
- Charles' law.
- Boyles law.

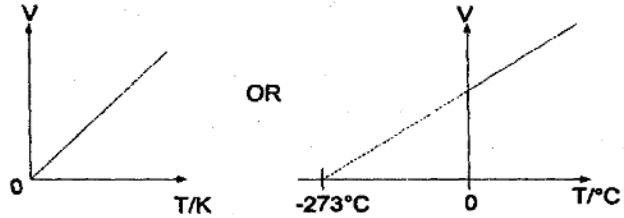
Pressure law

It states that the pressure P of a fixed mass of a gas at constant volume is directly proportional to its absolute temperature, *T*. That is, $P \propto T$



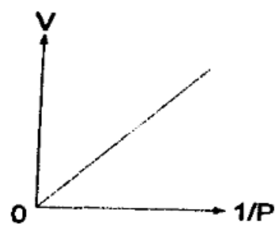
Charles' law

It states that the volume *V* of a fixed mass of a gas at constant pressure is directly proportional to its absolute temperature, *T*. That is $V \propto T$



Boyle's law

It states that the pressure P of a fixed mass of a gas at constant absolute temperature is inversely proportional to its volume, *V*. That is, $P \propto \frac{1}{V}$



IDEAL GASES

An ideal gas is a gas which perfectly obeys the gas laws under all conditions and its intermolecular forces are negligible.

NOTE: No gas is perfectly ideal since the behavior of gases change under varying conditions. Therefore, the concept of ideal gases is simply imaginary.

Characteristics of an ideal gas

- It obeys gas laws at all temperatures.
- There are no intermolecular forces between molecules of an ideal gas.
- Their molecules move with a constant velocity in between collisions.
- Their molecules make perfectly elastic collisions. Their molecules occupy negligible volumes compared to the volume of the gas.

Ideal gas equation

The pressure *P*, volume *V*, and absolute temperature *T*, are related by the expression

$$\frac{PV}{T} = constant$$

This expression is the ideal gas equation. The value of the constant depends on the mass of the gas and experiments with real gases at low enough pressures show that it has the same value R for all the gases if one mole is considered. R is called the universal molar gas constant and has a value $8.31Jmol^{-1}K^{-1}$

Therefore, for one mole, the equation becomes;

$$PV = RT$$

This is called the equation of state for an ideal gas, the state being determined by the values of pressure and temperature. Also, for n moles of an ideal gas,

$PV = nRT$

NOTE:

1. n is the number of moles of the gas and is equal to the ratio of the mass of the gas m (in kg) to its molar mass M (in $kgmol^{-1}$). Therefore, if M_r is the relative molecular mass of the gas, then

$$M = M_r \times 10^{-3}$$

$$thus, n = \frac{m}{M}$$

2. Avogadro constant/number is the number of particles in one mole of a gas and has a constant value 6.02×10^{23}. The particles may be atoms, electrons, e.t.c. Therefore, if N_A is the number of particles in a gas of n moles, then

$$N = nN_A$$

3. In $PV = RT$, V is the molar volume and is equal to the ratio of molar mass (in $kgmol^{-1}$) to density (in kgm^{-3}). Therefore, if ρ is the density of the gas, then

$$V = \frac{M}{\rho}(in\ m^3mol^{-1})$$

4. In $PV = nRT, the\ unit\ of\ P\ is\ Nm^{-2}, V\ is\ m^3, and\ T\ is\ K$

5. Using $PV = nRT$ for one mole of an ideal gas at s.t.p

$$P = 1.01 \times 10^5Nm^{-2}, V = 22.4 \times 10^3m^3, T = 273K, and\ n = 1$$

Thus

$$R = \frac{PV}{nT} = \left[\frac{(1.01\times10^5)(22.4\times10^3)}{1\times273}\right] \approx 8.31Jmol^{-1}K^{-1}$$

KINETIC PRESSURE OF AN IDEAL GAS

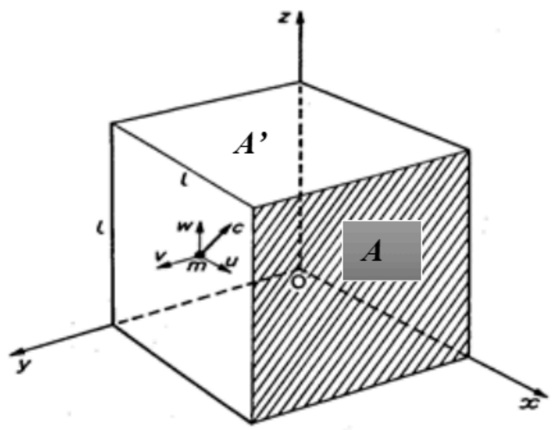
Assumptions made in derivation of kinetic pressure of an ideal gas

- Molecules exert no force on each other except during collisions,
- Collisions between the molecules and the container are perfectly elastic,
- The volume of the molecules is negligible compared to that occupied by the gas.

- The duration of a collision is negligible compared to the time spent between collisions.
- The number of molecules is large and they are in a state of continuous random motion.

Derivation of kinetic pressure of an ideal gas

Suppose we have a hollow cubical vessel of each side *l* with perfectly elastic walls, containing a very large number of molecules, say *N*. Consider motion of one molecule having a velocity c_1. The velocity can be resolved into three rectangular components u_1, v_1, w_1 parallel to X-axis, Y-axis and Z-axis respectively.



Consider the motion in the OX-direction to face A. The molecule strikes face A with a velocity u_1 and rebounds with the same velocity in the backward direction, since the collision is perfectly elastic. If *m* is the mass of the molecule,

The change in momentum = $[mu_1 - (-mu_1)] = 2mu_1$

The molecule now travels from face A to A', it strikes A' and travel back again to A after covering-a distance 2*l*

$$\begin{pmatrix}\text{time spent between} \\ \text{collisions}\end{pmatrix} = \left(\frac{\text{distance}}{\text{velocity}}\right) = \frac{2l}{u_1}$$

$$\begin{pmatrix}\text{rate of change of} \\ \text{momentum}\end{pmatrix} = \left(\frac{\text{change in momentum}}{\text{time spent between collisions}}\right)$$

$$= \frac{2mu_1}{\left(2l/u_1\right)} = \frac{mu_1^2}{l}$$

According to Newton's second law of motion, the rate of change of momentum is equal to the force exerted on the wall A. If F_1 is the force, then

$$F_1 = \frac{mu_1^2}{l}$$

For N molecules, the total force exerted is given by;

$$F = \frac{m}{l}({u_1}^2 + {u_2}^2 + \cdots \dots \dots \dots . + {u_N}^2)$$

Total pressure exerted on wall A is given by;

$$P = \left(\frac{total\ force}{cross-sectional\ area}\right) = \frac{\left[\frac{m}{l}({u_1}^2 + {u_2}^2 + \cdots \dots \dots + {u_N}^2)\right]}{l^2}$$

$$P = \frac{m}{l^3}({u_1}^2 + {u_2}^2 + \cdots \dots \dots \dots . + {u_N}^2) \dots \dots \dots \dots . (i)$$

Suppose $\overline{u^2}$ is the mean square value of all the velocities in the OX – direction, it implies that;

$$\overline{u^2} = \frac{({u_1}^2 + {u_2}^2 + \cdots \dots \dots + {u_N}^2)}{N}$$

$$({u_1}^2 + {u_2}^2 + \cdots \dots \dots \dots . + {u_N}^2) = N\overline{u^2} \dots \dots \dots (ii)$$

Substituting equation (ii) in (i) gives;

$$P = \frac{m}{l^3}\left(N\overline{u^2}\right) \quad \dots\dots\dots\dots\text{(iii)}$$

Since u, v, w are components of the velocity in the X, Y, Z directions and c is the resultant velocity of the three components, then

$$\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}$$

For a large number of molecules, it is assumed that $\overline{u^2} = \overline{v^2} = \overline{w^2}$

Thus $\overline{c^2} = 3\overline{u^2}$ *implying that* $\overline{u^2} = \frac{1}{3}\overline{c^2}$ ……..(iv)

Substituting equation (iv) in (iii) gives;

$$P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$$

If ρ is the density of the gas, then

$$\rho = \left(\frac{total\ mass\ of\ the\ gas\ molecules}{volume\ of\ the\ cube}\right) = \frac{Nm}{l^3}$$

Thus, $P = \frac{1}{3}\rho\overline{c^2}$

This is the expression of the kinetic pressure of an ideal gas.

Derivation of Avogadro's law from kinetic pressure

It states that, equal volumes of ideal gases, existing under the same conditions of temperature and pressure, contain equal numbers of molecules.

Consider two gases 1 and 2. From the expression of their kinetic pressures, it implies that;

$$P_1V_1 = \frac{1}{3}(N_1m_1)\overline{{c_1}^2} = nRT_1 \quad and \quad P_2V_2 = \frac{1}{3}(N_2m_2)\overline{{c_2}^2} = nRT_2$$

If their pressures, temperatures, and volumes are the same, then

$$P_1 = P_2\,, T_1 = T_2 \quad and\ V_1 = V_2$$

Thus $$P_1V_1 = P_2V_2$$

$$\frac{1}{3}(N_1m_1)\overline{c_1{}^2} = \frac{1}{3}(N_2m_2)\overline{c_2{}^2}$$

$$\frac{2}{3}N_1\left(\frac{1}{2}m_1\overline{c_1{}^2}\right) = \frac{2}{3}N_2\left(\frac{1}{2}m_2\overline{c_2{}^2}\right)$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

Then $\frac{2}{3}N_1\left(\frac{3}{2}K_BT_1\right) = \frac{2}{3}N_2\left(\frac{3}{2}K_BT_2\right)$

Also since $T_1 = T_2 = T$

$$\frac{2}{3}N_1\left(\frac{3}{2}K_BT\right) = \frac{2}{3}N_2\left(\frac{3}{2}K_BT\right)$$

$$N_1 = N_2$$

This is the expression of Avogadro's law

Deduction of Dalton's law of partial pressure from kinetic pressure

It states that in a mixture of ideal gases, the total pressure is equal to the sum of the partial pressures due to each gas in the container.

Consider two gases 1 and 2 .From the expression of their kinetic pressures, it implies that

$$P_1V_1 = \frac{1}{3}(N_1m_1)\overline{c_1{}^2} \quad and\ \ P_2V_2 = \frac{1}{3}(N_2m_2)\overline{c_2{}^2}$$

By definition, **partial pressure** of a gas is that pressure that a gas would exert if it alone occupies the container.

It therefore implies that if V is the total volume of the container, then $V_1 = V_2 = V$. Thus,

$$P_1V = \frac{1}{3}(N_1m_1)\overline{c_1{}^2} \quad and\ \ P_2V = \frac{1}{3}(N_2m_2)\overline{c_2{}^2}$$

Adding the two equations gives;

$$P_1V + P_2V = \frac{1}{3}(N_1m_1)\overline{c_1{}^2} + \frac{1}{3}(N_2m_2)\overline{c_2{}^2}$$

$$(P_1 + P_2)V = \frac{2}{3}\left(\frac{1}{2}N_1m_1\right)\overline{c_1{}^2} + \frac{2}{3}\left(\frac{1}{2}N_2m_2\right)\overline{c_2{}^2}$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

Then $(P_1 + P_2)V = \frac{2}{3}N_1\left(\frac{3}{2}K_BT_1\right) + \frac{2}{3}N_2\left(\frac{3}{2}K_BT_2\right)$

Also since $T_1 = T_2 = T$, then

$$(P_1 + P_2)V = \frac{2}{3}N_1\left(\frac{3}{2}K_BT\right) + \frac{2}{3}N_2\left(\frac{3}{2}K_BT\right)$$

$$(P_1 + P_2)V = \frac{2}{3}(N_1 + N_2)\left(\frac{3}{2}K_BT\right)$$

$$(P_1 + P_2)V = \frac{2}{3}(N_1 + N_2)\left(\frac{1}{2}m\overline{c^2}\right)$$

But $N_1 + N_2 = (total\ number\ of\ molecules) = N$

Therefore, $(P_1 + P_2)V = \frac{1}{3}Nm\overline{c^2}$

If P is the total pressure of the gas mixture, then

$$\frac{1}{3}Nm\overline{c^2} = PV$$

Thus , $(P_1 + P_2)V = PV$

Therefore $P_1 + P_2 = P$

This is the expression for Dalton's law of partial pressures

Deduction of Graham's law of diffusion from kinetic pressure

It states that the rate of diffusion of a gas is inversely proportional to the square root of its density under the same conditions of temperature and pressure.

Thus, from $P = \frac{1}{3}\rho\overline{c^2}$

Since $P_1 = P_2, then$

$$\frac{1}{3}\rho_1\overline{c_1}^2 = \frac{1}{3}\rho_2\overline{c_2}^2$$

$$\frac{\overline{c_1^2}}{\overline{c_2^2}} = \frac{\rho_1}{\rho_2}$$

$$\frac{rate\ of\ diffusion\ of\ gas\ 1}{rate\ of\ diffusion\ of\ gas\ 2} = \sqrt{\left(\frac{\overline{c_1^2}}{\overline{c_2^2}}\right)} = \sqrt{\left(\frac{\rho_1}{\rho_2}\right)}$$

This is the expression for Graham's law of diffusion

Variation of root mean square speed

From $P = \frac{1}{3}\rho\overline{c^2}$

$$PV = \frac{1}{3}(Nm)\overline{c^2}$$

But molar mass of a gas = Nm = M

Therefore $PV = \frac{1}{3}M\overline{c^2}$

Also $PV = RT$, thus $RT = \frac{1}{3}M\overline{c^2}$

$$\overline{c^2} = \frac{3RT}{M}$$

Therefore, the root mean square speed is given by

$$\sqrt{\overline{c^2}} = \sqrt{\frac{3RT}{M}}$$

In summary, $\overline{c^2} \propto T$, $\overline{c^2} \propto \frac{1}{\rho}$, $\overline{c^2} \propto \frac{1}{m}$

NOTE:
Root mean square speed is different from average sped. Root mean square speed is the measure of the rate of diffusion of a gas. It is the square root of the mean square values of molecular speed. Average speed on the other hand is the mean velocity of gas molecules.

Using the expression of kinetic pressure to derive the ideal gas equation

$$P = \frac{1}{3}\rho\overline{c^2}$$

$$P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$$

$$l^3 = (volume\ of\ a\ gas) = V$$

Therefore, $PV = \frac{1}{3}(Nm)\overline{c^2}$

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right)$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

$$PV = \frac{2}{3}N\left(\frac{3}{2}K_BT\right)$$

$$PV = NK_BT$$

But $N = nN_A$ $and\ K_B = \frac{R}{N_A}$

Therefore, $PV = (nN_A)\left(\frac{R}{N_A}\right)T$

$$PV = nRT$$

This is the expression for the ideal gas equation

Using the expression for kinetic pressure to derive Boyle's law

From $P = \frac{1}{3}\rho\overline{c^2}$

$$P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$$

$$l^3 = (volume\ of\ a\ gas) = V$$

Therefore, $PV = \frac{1}{3}(Nm)\overline{c^2}$

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right)$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

$$PV = \frac{2}{3}N\left(\frac{3}{2}K_BT\right)$$

$$PV = NK_BT$$

K_B is a constant. For a fixed mass of a gas, N is constant. Therefore, if temperature T is constant, then

$$PV = constant$$

Using the expression for kinetic pressure to derive Charles's law

From $P = \frac{1}{3}\rho\overline{c^2}$

$$P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$$

$$l^3 = (volume\ of\ a\ gas) = V$$

Therefore, $PV = \frac{1}{3}(Nm)\overline{c^2}$

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right)$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

$$PV = \frac{2}{3}N\left(\frac{3}{2}K_BT\right)$$

$$PV = NK_BT$$

K_B is a constant. For a fixed mass of a gas, N is constant. Therefore, if pressure P is constant, then

$$\frac{V}{T} = constant$$

Using the expression for kinetic pressure to derive pressure law

From $P = \frac{1}{3}\rho\overline{c^2}$

$$P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$$

$$l^3 = (volume\ of\ a\ gas) = V$$

Therefore, $PV = \frac{1}{3}(Nm)\overline{c^2}$

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right)$$

Since $K.E = \left(\frac{1}{2}m\overline{c^2}\right) = \left(\frac{3}{2}K_BT\right)$

$$PV = \frac{2}{3}N\left(\frac{3}{2}K_BT\right)$$

$$PV = NK_BT$$

K_B is a constant. For a fixed mass of a gas, N is constant. Therefore, if volume V is constant, then

$$\frac{P}{T} = constant$$

Relationship between kinetic energy of a molecule and temperature of a gas

From $P = \frac{1}{3}\left(\frac{Nm}{l^3}\right)\overline{c^2}$

$$l^3 = (volume\ of\ a\ gas) = V$$

Therefore, $PV = \frac{1}{3}(Nm)\overline{c^2}$

But $N = nN_A$. Therefore, $3PV = nN_Am\overline{c^2}$. Also $PV = nRT$,

Thus $3nRT = nN_Am\overline{c^2}$

$$m\overline{c^2} = 3\left(\frac{R}{N_A}\right)T$$

Now the average kinetic energy (K.E) of translational motion per molecule is given by;

$$K.E = \frac{1}{2}m\overline{c^2} = \frac{1}{2}\left[3\left(\frac{R}{N_A}\right)T\right] = \frac{3}{2}\left(\frac{R}{N_A}\right)T$$

The ratio $\frac{R}{N_A}$ is called the Boltzmann's constant (K_B). It is the constant per molecule. That is;

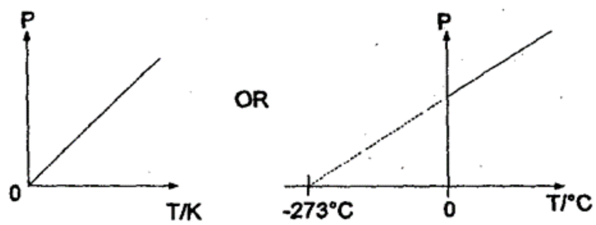
$$K.E = \frac{3}{2}K_BT$$

Therefore, $K.E \propto T$

EXPLANATION OF THE GAS LAWS USING KINETIC THEORY

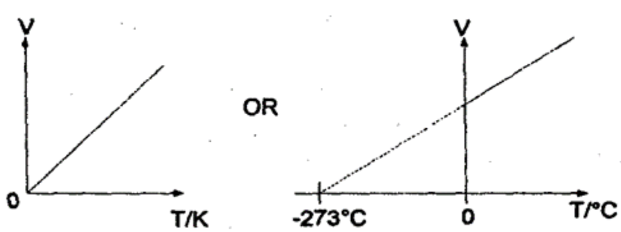
The theory considers molecules of a gas to be like elastic spheres. Each time one of the molecules strikes the wall of the container, it rebounds. The force produced on the wall by a molecule is the momentum change per second. So the gas pressure due to the bombarding molecules is proportional to their average total momentum per second; normal to the wall.

Pressure law (effect of temperature on a gas at constant volume)



Pressure of a gas is due to is due to the rate of change of momentum of the gas molecules on collision with the wall of the container.
From kinetic theory, gas molecules have an average speed which increases with temperature. Therefore, when temperature increases, the kinetic energy of molecules increases; making molecules move at a high speed.
Since the volume of the gas is constant, the rate of change of momentum at the walls of the container will increase due to increased rate of collisions of the molecules with the walls. Thus, the pressure of the gas increases with increase in temperature.

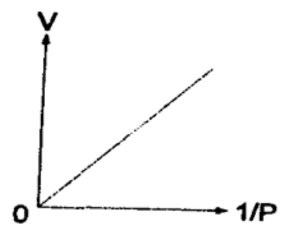
Charles' law (effect of temperature on a gas at constant pressure)



If pressure of a gas is to remain constant, the rate of change of momentum of molecules on collision with the walls must also remain constant.
From kinetic theory, gas molecules have an average speed which increases with temperature. Therefore, when temperature increases, the kinetic energy of molecules increases; making molecules move at a high speed.
Since the rate of change of momentum at the walls is to remain constant, the volume of the gas

must increase so that molecules travel farther (spend more time) between collisions with the walls.

Boyle's law (effect of pressure on a gas at constant temperature)



From kinetic theory, gas molecules have an average speed which increases with temperature. Therefore, if temperature of a gas is to remain constant, the average speed of molecules must also remain constant.

When the pressure of a gas increases, the rate of change of momentum of molecules on collision with the walls also increases, since the average speed of molecules is to remain constant, the volume of the gas must decrease so that molecules spend less time between collisions with the walls.

REAL GASES

A real gas is one in which:

- The intermolecular forces of attraction exist.
- The volume of molecules is not negligible compared to that occupied by the gas.
- Doesn't obey Boyle's law. It obeys a law derived from the Vander Waal's equation.

Vander Waal's equation

This is an equation that represents the behavior of real gases. It's a modification of the ideal gas equation *[PV = RT]* and it takes into account the finite size of the molecules and the attractive forces between them.

To obtain this equation, two of the assumptions made in the simple kinetic theory of ideal gases (the ones stated under "derivation of kinetic pressure of an ideal gas") are modified.

These modifications include;

- The volume of the molecules isn't negligible compared to that occupied by the gas.
- The attractive forces between the molecules are not negligible. With these modifications, the Vander Waal's equation is

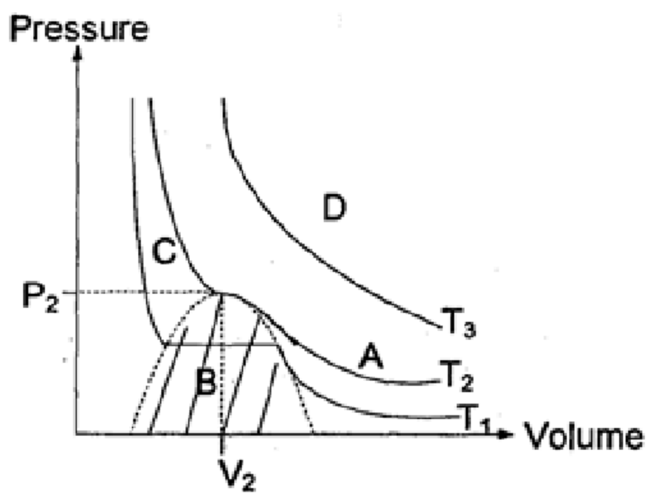
$$\left(P + \frac{a}{V^2}\right)(V - b) = constant$$

Accounting for the terms $\frac{a}{V^2}$ *and b in the equation*

Because of the intermolecular force of attraction, any molecule approaching the walls of the container exerts less pressure than the pressure in the bulk (interior). This accounts for a pressure defect

The volume of the molecules may not be negligible compared to the volume of the container occupied by the gas. Therefore, the co-factor *b* accounts for the finite volume of the molecules themselves. Therefore, *(V − b)* is the free volume of movement.

***P* - *V* sketch graph indicating the isotherms (different states) of a real gas at different temperatures**



A = Unsaturated vapour.

B = Liquid and saturated vapour.

C = Liquid state.

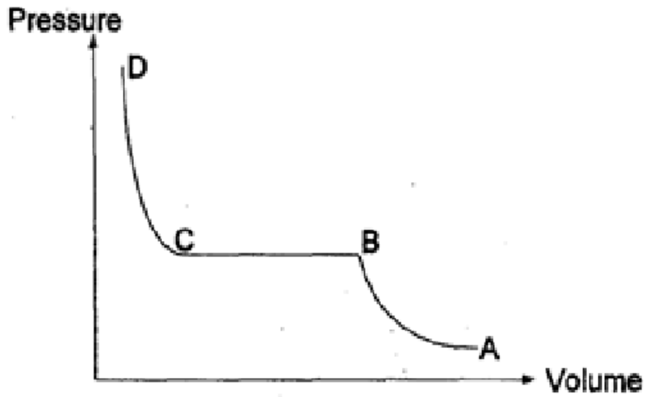
D = Gaseous state.

The graph above shows that gases can only liquefy if compressed at temperatures below the critical temperature. In the figure above, T_2 is the critical temperature.

Isotherm at T_1 shows the behavior of a real gas at temperatures below the critical temperature, (it can liquefy when compressed).

Isotherm at T_3, shows the behavior of a real gas at temperatures above the critical temperature, (it obeys Boyle's law and can't be liquefied)

P - V sketch graph for a real gas undergoing below its critical temperature



At A, the substance is in gaseous phase. In region AB, there's unsaturated vapour which fairly obeys Boyle's law. That is, as pressure increases gradually, volume also decreases according to Boyle's law.

At B, liquefaction begins. In region BC, the vapour is saturated. That: is, pressure remains constant as volume reduces. At C, liquefaction is complete. In the region CD, the entire vapour has turned into liquid and there's a very small change in volume for a fairly large increase in pressure. This is because liquids are incompressible.

NOTE:

(1) Critical temperature (T_2) is the maximum temperature required to liquefy a gas. OR critical temperature is the temperature above which *a* gas can't be liquefied no matter how great the pressure may be.

(2) Critical pressure (P_2) is the minimum pressure required to liquefy a gas at its critical temperature.

(3) Critical volume (V_2) is the volume occupied by one mole of a gas at its critical temperature.

Conditions under which real gases behaves as ideal gases

- **High temperatures**: at high temperatures, the forces of attraction for real gases are so weak such that they become negligible thus behaving like ideal gases.
- **Very low pressures**: at very low pressures, the gas molecules in a particular container will be few and very much farther apart. This implies that the volume of gas molecules becomes negligible compared to the volume of the container.

Differences between real and ideal gases

Real gases	Ideal gases
Intermolecular forces exist and are significant	Intermolecular forces are negligible
Volume of its molecules is significant compared to the volume of the gas	Volume of its molecules is negligible compared to the volume of the gas
Velocity of its molecules is not constant due to intermolecular forces	Velocity of its molecules is constant since intermolecular forces are negligible
Doesn't obey boyle's law	Obey Boyle's law

WORKED EXAMPLES

Where applicable, use the following constants

Avogadro's number $N_A = 6.02 \times 10^{23} mol^{-1}$

Gas constant, $R = 8.31 Jmol^{-1}K^{-1}$

1. A container of a gas of volume $1 \times 10^{-3} m^3$ contains helium of pressure 2×10^5Pa when the temperature is 300K.

i. What is the mass of helium in the container?

ii. How many helium atoms are there in the container?

iii. Calculate the root mean square speed of the helium atoms. Take relative atomic mass of helium = 4.

Solution

$P = 2 \times 10^5 Pa,\ V = 1 \times 10^{-3} m^3, T = 300K, M = 4n$

(i) From $PV = nRT$

Number of moles of helium in the container

$$n = \frac{PV}{RT} = \frac{2\times10^5\times1\times10^{-3}}{8.31\times300} = 0.08\, moles$$

The mass of helium in the container is

$$m = nM = 0.08 \times 4 = 0.32\text{g}$$

(ii)The number of helium atoms in the container are

$$N = nN_A = 0.08 \times 6.02 \times 10^{23} = 4.816 \times 10^{22} atoms$$

(iii)The density of helium atoms in the container is

$$\rho = \frac{m}{V} = \frac{0.32\times10^{-3}}{1\times10^{-3}} = 0.32 kgm^{-3}$$

Therefore, from $P = \frac{1}{3}\rho\overline{c^2}$

$$\overline{c^2} = \frac{3P}{\rho} = \frac{3\times2\times10^5}{0.32} = 1.875 \times 10^6$$

The root mean square speed of the helium atoms is

$$\sqrt{\overline{c^2}} = \sqrt{1.875 \times 10^6} = 1.369 \times 10^3 ms^{-1}$$

2. The root mean square speed of hydrogen molecules at s.t.p is $1.84 \times 10^3 ms^{-1}$. Find the root mean square speed of hydrogen at 100°C.

Solution

$$\sqrt{\overline{c_1{}^2}} = 1.84 \times 10^3 ms^{-1}, T_1 = 273K, T_2 = 100 + 273K = 373K, \sqrt{\overline{c_2{}^2}} = ?$$

$$\overline{c^2} \propto T$$

$$\frac{\overline{c_1{}^2}}{\overline{c_2{}^2}} = \frac{T_1}{T_2}$$

$$\sqrt{\overline{c_2{}^2}} = \left(\sqrt{\frac{T_1}{T_2}}\right) \times \sqrt{\overline{c_1{}^2}}$$

$$\sqrt{\overline{c_2{}^2}} = \left(\sqrt{\frac{373}{273}}\right) \times 1.84 \times 10^3 = 2.151 \times 10^3 ms^{-1}$$

3. Calculate the root mean square speed of nitrogen molecules at 300K given that the molecular mass of nitrogen is 0.028kg per mole.

Solution

T = 300K, R = $8.31 Jmol^{-1}K^{-1}$, M = 0.028kg

$$\sqrt{\overline{c^2}} = \sqrt{\left(\frac{3RT}{M}\right)}$$

$$\sqrt{\overline{c^2}} = \sqrt{\left(\frac{3\times8.31\times300}{0.028}\right)} = 5.168 \times 10^2 ms^{-1}$$

4. Calculate the root mean square speed of the speed of the molecules of an ideal gas at 127°C given that the density of the gas at a pressure of 1×10^5Pa and a temperature of 0°C is 1.43 kgm^{-3}

Solution

$P_1 = 1 \times 10^5 Pa$, $\rho_1 =$ 1.43 kgm^{-3}

$T_1 = 0 + 273 = 273K, T_2 = 127 + 273 = 400K$

From $P = \frac{1}{3}\rho\overline{c^2}$

$\overline{c_1{}^2} = \frac{3P_1}{\rho_1} = \frac{3\times1\times10^5}{1.43} = 2.098\times10^5$

$\overline{c^2} \propto T$

$\frac{\overline{c_1{}^2}}{\overline{c_2{}^2}} = \frac{T_1}{T_2}$

$\sqrt{\overline{c_2{}^2}} = \left(\sqrt{\frac{T_1}{T_2}}\right)\times\sqrt{\overline{c_1{}^2}}$

$\sqrt{\overline{c_2{}^2}} = \sqrt{\frac{400}{273}}\times\sqrt{2.098\times10^5} = 5.544\times10^2 ms^{-1}$

Trial questions

1. Air consists of 20% oxygen and 80% nitrogen. The relative molecular mass of oxygen and nitrogen are 32 and 28 respectively.

(i) Calculate the ratio of the mass of oxygen to nitrogen.

(ii) Calculate the ratio of the partial pressure of oxygen to nitrogen in air.

(iii) Explain why oxygen and nitrogen are gases found in the atmosphere close to the earth.

[Ans: 7:8, 1:4]

2. One mole of an ideal gas at 300K is subjected to a pressure of 1 x 10^5 *Pa* and its volume is $0.025m^3$. Find

i. Molar gas constant,

ii. Boltzmann's constant.

iii. Average kinetic energy of the molecules of the gas.

[Ans: 8.33 $Jmol^{-1}K^{-1}$, $1.384\times10^{-23}JK^{-1}$, $6.23\times10^{-21}J$]

3. A beam of 2 x 10^{22} nitrogen atoms, each of mass 2.32 x $10^{-26}kg$ is incident normally on a wall of cubical container of edge 10cm. The beam is reflected through 180°. If the mean speed of the atoms is $480ms^{-1}$, find the pressure exerted by the nitrogen gas.

[Ans: $1.014\times10^5 Pa$]

4. Calculate of sound in the atmosphere of Jupiter, knowing that its main constituents is methane vapour and that the speed of sound in gas is 0.682 times the root mean square speed of the gas molecules.(Temperature of Jupiter atmosphere = 130°Cand molecular mass of methane = 16.06g/mol) [Ans: 530.2 ms^{-1}]

5. Molecules of gas X have a root mean square speed of $900 ms^{-1}$ at 27°C and 1×10^5Pa. Calculate the root mean square speed at 127°C with the pressure kept constant.

[Ans: $1039\ ms^{-1}$]

CHAPTER 18: VAPOURS

TERMS AND DEFINITIONS

(a) Vapour: This is a state of matter that can be turned into liquid by application of pressure alone. Vapour can also be defined as a molecular substance which is in gaseous phase below its critical temperature.

(b) Gas: This is a state of matter that can't be turned into liquid by application of pressure alone. Gas can also be defined as a molecular substance which is in gaseous phase above its critical temperature.

(c) Saturated vapour: A saturated vapour is one which is in dynamic equilibrium with its own liquid. Unsaturated vapour: An unsaturated vapour is one which is in dynamic equilibrium with its own liquid.

(d) Saturated vapour pressure: This is the pressure exerted by a vapour which is in a state of dynamic equilibrium with its own liquid.

(e) Unsaturated vapour pressure: This is the pressure exerted by a vapour which is not in dynamic equilibrium with its own liquid.

EVAPORATION

This is the process by which a liquid becomes vapour. It takes place at all temperatures but occurs at the greatest rate when the liquid is at its boiling point.

Factors that determine the rate of evaporation

- Temperature. Increasing the temperature of a liquid increases the rate of evaporation
- Surface area. Increasing the surface area of the liquid increases the surface area of the liquid
- Draught or wind blowing over the surface of the liquid. The rate of evaporation increases when there's too much wind blowing since the vapour molecules are removed before they return to the liquid.

Kinetic theory of evaporation

This explains why evaporation causes cooling.

Molecules of a liquid are always in a state of continuous random motion and make frequent collisions with each other and with the walls of the container.

If a molecule near the surface of the liquid gains enough energy to overcome the attractive forces of the molecules below it, it escapes from the surface. The process is known as evaporation.

Therefore, when a liquid evaporates, it loses most of its molecules with a greater kinetic energy and the ones that remain have low kinetic energies.

Since temperature is proportional to the average kinetic energy of molecules, the decrease in kinetic energy of the molecules in the liquid decreases the temperature of the liquid thus the liquid cools.

Why it's much cooler in the valley than on top of hills at night

At night, air loses heat and its density increases. Therefore, the more denser air, which is cool, tends to move down to the deeper areas like valleys while the less dense air, which is warm, stays on top like hills. This makes valleys to be cooler than hills at night.

Why a baby has to be wrapped on a cold day

On a cold day, the temperature of the body is higher than that of the surroundings. Therefore, there's a net rate of heat loss from the body which depends on the temperature difference between the body and the surrounding. Therefore, a baby should be wrapped to avoid loss of heat from the baby to the surrounding.

Why a bottle of soda left in a freezer overnight breaks

There's a high percentage of water in soda. Therefore, until the temperature decreases up to 4^0C, it still behaves like any other liquid.

However, when the temperature exceeds 4°C, it begins expanding due to anomalous expansion of water. This expansion makes the bottle of soda to break.

Why cloudy nights are warmer than cloudless nights

At night, radiational cooling usually occurs. This is the cooling of the earth's surface and the air near the surface and it’s caused by infrared radiation from the earth's surface and from the

atmosphere.
The clouds therefore act like *a* blanket and this blanketing effect prevents heat from escaping through the atmosphere and go into space. Thus cloudy nights are warmer than cloudless nights

DEW FORMATION

Dew point

This is the temperature at which water vapour, present in the air, is just sufficient to saturate it.

Conditions for dew formation

- Low temperatures.
- Humidity (amount of water vapour in the air),

Dew formation on grass

During the night, air temperature falls as the earth cools. If the temperature falls below the dew point, water condenses from air and forms dew.

BOILING

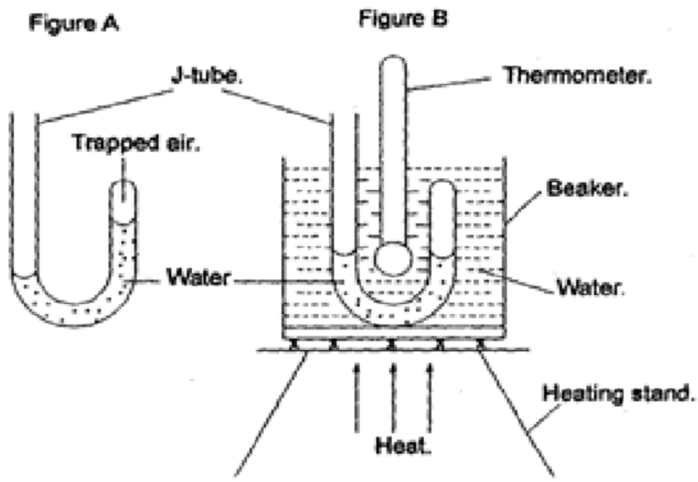
Boiling point is the temperature at which saturated vapour pressure equals external pressure.

Effect of altitude on boiling

Altitude is the height above the earth's surface. An increase in the height above the earth's surface decreases the weight and hence the density of the air above. This in turn decreases the atmospheric pressure. Thus, atmospheric pressure decreases with increase in altitude.

Experiment to show that a liquid boils only when its saturated vapour pressure equals external pressure

Some water is placed in a J-tube such that some air is trapped by the liquid column as shown in figure (A). The J-tube is then placed in a beaker of water subjected to heat as shown in figure (B).
The air in the J-tube will be observed to remain trapped as before until the water in the beaker starts boiling. At boiling point, the water in the J-tube then comes to the same level in each limb. This implies that the vapour pressure in the closed limb equals to the external pressure.



Therefore, a liquid boils only when its saturated vapour pressure is equal to the external pressure.

Why it's possible to make water boil at temperatures below its boiling point

It's possible to make water boil at temperatures below its boiling point because boiling of water occurs only when its saturated vapour pressure equals the atmospheric pressure. Since external/atmospheric pressure varies with altitude, it implies that when altitude is increased, the atmospheric pressure will reduce.

Thus, the saturated vapour pressure equals the lowered external pressure easily and since pressure is directly proportional to temperature, boiling takes place before the normal boiling point.

How cooking of water at 76cmHg and 100°C is achieved

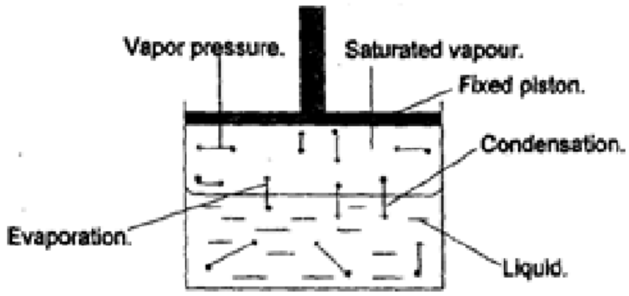
Cooking utensils/pressure cookers have lids which prevent steam from escaping. As the water inside the cooking utensil is being heated, density of steam above the water increases which in turn increases the steam pressure above the water.

At 100°C the steam pressure equals 76cmHg and hence, the water boils. If heating is continued, steam pressure may rise above 76cmHg hence causing explosion of the system.

Therefore, to ensure cooking at 76cmHg and 100^0C, the lid should have a safety valve designed such that it opens when pressure inside the utensil is 76cmHg.

SATURATED VAPOUR PRESSURE

Occurrence of saturated vapour pressure



Consider a liquid heated in a closed container. The most energetic molecules overcome attraction by other molecules and leave the surface of the liquid to become vapour molecules. This is process known as evaporation.

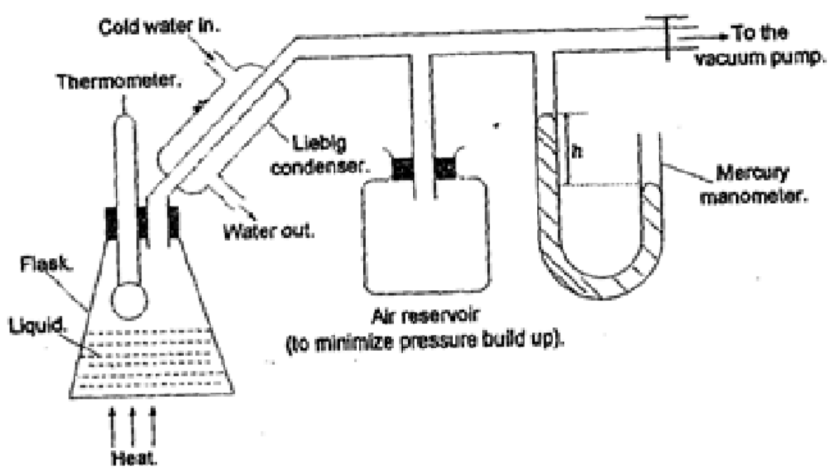
Vapour molecules collide with the walls of the container giving rise to vapour pressure. They lose the energy they had and re-enter the liquid. This process is known as condensation.

If the rate of evaporation is equal to the rate of condensation, dynamic equilibrium is said to be attained. The density of the vapour and hence the vapour pressures is maximum at that temperature and this pressure is saturated vapour pressure.

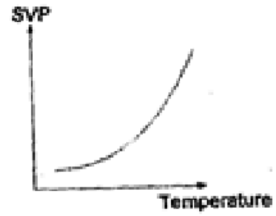
Experiment to determine the variation of saturated vapour pressure with temperature

The apparatus is set up as shown below. The liquid in the flask is raised to a temperature 6 and then the pressure is reduced to a value when the liquid starts boiling. The manometer reading h is taken. Thus, the saturated vapour pressure $S.\ V.\ P = (H - h)$.

Where H is atmospheric pressure.



The experiment is repeated for different temperatures and the corresponding saturated vapour pressure *S. V. P* calculated. A graph of saturated vapour pressure *S. V. P* and temperature is plotted as below.

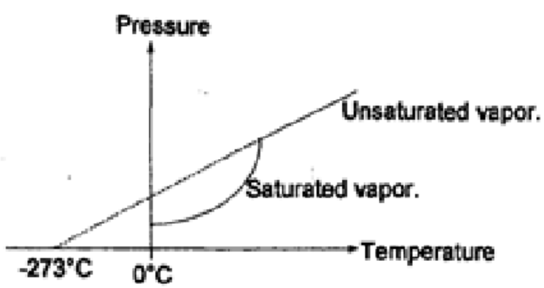


The graph shows that saturated vapour pressure increases with temperature but not linearly.

Effect of temperature on saturated vapour pressure at constant volume

For a saturated vapour, the rate of evaporation equals the rate of condensation. When temperature increases, the mean kinetic energy of the molecules increases and the most energetic ones leave the liquid. This increases the rate of evaporation. The increase in the rate of evaporation increases the density of vapour above the liquid and this increases the rate of condensation. Dynamic equilibrium is obtained at a higher value thus increasing saturated vapour pressure. The increase in saturated vapour pressure is almost exponential.

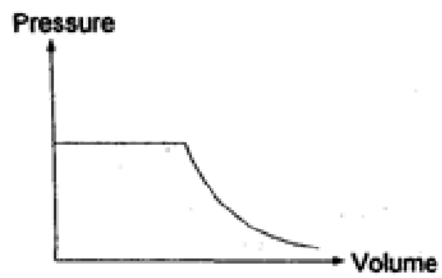
However, if the temperature is increased further, a point is reached where all the liquid evaporates and then unsaturated vapour is formed which behaves like an ideal gas. That is, pressure increases linearly with temperature.



Effect volume on saturated vapour pressure at constant temperature

When volume is increased, there's a momentarily decrease in the vapour density which reduces the rate of condensation and increases the rate of evaporation hence restoring the pressure to its original value. Therefore, saturated vapour pressure is independent of volume increase.

However, if the volume is increased further, a point is reached where all the liquid evaporates and then unsaturated vapour is formed which behaves like an ideal gas. That is, pressure decreases exponentially with increase in volume.



Why saturated vapour pressure is not affected by a decrease in volume at constant temperature

When the volume of the container is reduced, the space above the liquid reduces. This increases the vapour density, rate of condensation and saturated vapour pressure. Since temperature is constant, the rate of evaporation remains constant. However, the increase in the rate of condensation decreases vapour density which causes a decrease in the rate of condensation and saturated vapour pressure. This makes dynamic equilibrium to be restored to the original value. Thus, volume change has no effect on saturated vapour

Differences between saturated and unsaturated vapour

Saturated vapour	Unsaturated vapour
It doesn't obey gal laws.	It obeys gas laws.
It's in dynamic equilibrium with its own liquid	It's not in dynamic equilibrium with its own liquid
It can only exist when the liquid is present	It doesn't need the present of a liquid.
It exits at a fixed temperature.	It exists at any temperature

Solving problems on vapours and mixtures

Unsaturated vapours obey gas laws while saturated vapours don't obey gas laws. If the vapour is just saturated at a temperature θ, it implies that beyond this temperature, the vapour is unsaturated and therefore obeys gas laws. However, for temperatures less than θ, the vapour is saturated and therefore doesn't obey gas laws. Furthermore, a temperature equal to θ is known as the saturation temperature.

(a) A mixture of gas and unsaturated vapour

A mixture of gas and unsaturated vapour obeys gas laws fairly well, each exerting the same pressure as if it alone occupied the total volume. Therefore, if P_g is the pressure of the gas, and P_u is the pressure of the unsaturated vapour in the mixture, it implies that the total pressure P_T, which obeys gas laws is

$$P_T = (P_g + P_u)$$

If V is the volume of the container containing the mixture and is at a temperature T, it implies that from the ideal gas equation,

$$\frac{P_T V}{T} = constant$$

(b) A mixture of gas and saturated vapour

Since saturated vapours don't obey gas laws, the ideal gas equation is applied on only the gas in the mixture. Therefore, if P_T is the total pressure of the mixture, and is the pressure of the saturated vapour in the mixture, it implies that the pressure of the gas, Pg which obeys gas laws, can be got from $P_T = (P_g + P_s)$

That is,

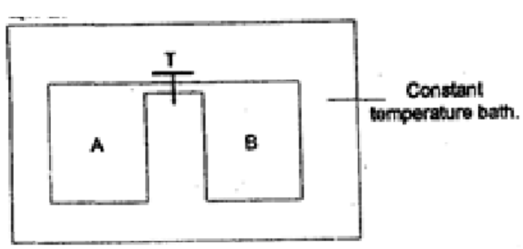
$$P_g = P_T - P_s$$

If V is the volume of the container containing the mixture and is at a temperature T, it implies that from the ideal gas equation,

$$\frac{P_g V}{T} = constant$$

WORKED EXAMPLES

1.



Two cylinders A and B, each of volume 1.5 litres, are joined in the middle by tap T and placed in a constant temperature bath at 60°C as shown. A contains a vacuum while B contains air and saturated vapour. The total pressure in B is 200mmHg. When tap T is opened, equilibrium is reached with water vapour remaining saturated. If the final pressure in the cylinder is

150mmHg, calculate the saturated vapour pressure of water at 60°C.

Solution

Let P_s be the saturated vapour pressure of water at 60°C.

Considering tap T closed

$$for\ A,\ \ V_1 = 0m^3, T_1 = 60 + 273 = 333K, P_1 = ?$$

$$for\ B,\ \ V_2 = 1.5 \times 10^3 m^3, T_2 = 60 + 273 = 333K, P_2 = 200 - P_s$$

$$from\ PV = nRT$$

$$n_1 = \frac{P_1V_1}{RT_1} \quad and\ \ n_2 = \frac{P_2V_2}{RT_2}$$

When tap T is opened;

$$V_1 = V_2 = 1.5 \times 10^3 m^3,\ V = 2 \times\ 1.5 \times 10^3 m^3 = 3\ \times 10^3 m^3$$

$$T = 60 + 273 = 333K, P = 200 - P_s$$

$$n = \frac{PV}{RT}$$

$$\begin{pmatrix} number\ of\ moles \\ when\ T\ is\ opened \end{pmatrix} = \begin{pmatrix} number\ of\ moles \\ when\ T\ is\ closed \end{pmatrix}$$

$$n = n_1 + n_2$$

$$\frac{PV}{RT} = \frac{P_1V_1}{RT_1} + \frac{P_2V_2}{RT_2}$$

$$\frac{PV}{T} = \frac{P_1V_1}{T_1} + \frac{P_2V_2}{T_2}$$

$$\frac{(150-P_s)\times 3\times 10^3}{333} = \frac{P_1\times 0}{333} + \frac{(200-P_s)\times 1.5\times 10^3}{333}$$

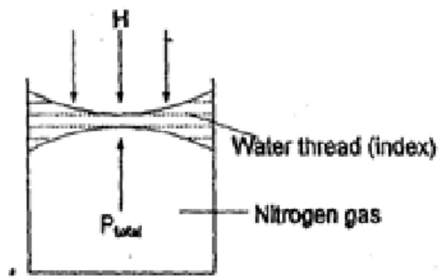
$$2(150 - P_s) = (200 - P_s)$$

$$P_s = 300 - 200 = 100mmHg$$

Note that when T is closed, $V_1 = 0m^3$ because A contains vacuum. However, when T is opened, the air and water vapour in B will occupy both cylinders thus, $V = 3 \times 10^3 m^3$. Also, note that P_s is constant because temperature is constant.

2. A column of nitrogen gas was trapped in a capillary tube of uniform cross-section area and closed at one end by a thread of water as shown below. The length of nitrogen column is 20.8cm at 25°C and 25.2cm at 83.3°C if its value at 25°C is 1.7 x 10^3Nm^{-2}. Neglect the weight of the water thread and assume atmospheric pressure, H remains constant at 1.01 x 10^5 Nm^{-2}.

Solution



Since the weight of the water thread is negligible, $P_{total} = H = 1.01 \times 10^5 Nm^{-2}$

At 25⁰C,

$l_1 = 20.8cm, V_1 = Al_1 = 20.8A\ cm^3, T_1 = 25 + 273 = 298K$

$H = 1.01 \times 10^5 Nm^{-2},\ \ P_{s1} = 1.7 \times 10^3 Nm^{-2}$

$P_1 = (H - P_{s1}) = (1.01 \times 10^5 - 1.7 \times 10^3) = 9.93 \times 10^4 Nm^{-2}$

At 25⁰C,

$l_2 = 25.2cm, V_2 = Al_2 = 25.2A\ cm^3, T_2 = 83.3 + 273 = 356.3K$

$H = 1.01 \times 10^5 Nm^{-2},\ \ P_{s2} =?\ P_1 = (H - P_{s2}) = (1.01 \times 10^5 -\ P_{s2})$

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

$$\frac{9.93\times10^4\times20.8A}{298} = \frac{(1.01\times10^5 - P_{s2})\times25.2A}{356.3}$$

$6.931 \times 10^3 = 7.143 \times 10^3 - (7.073 \times 10^{-2})P_{s2}$

$P_{s2} = 2.997 \times 10^3\ Nm^{-2}$

Note that what is trapped in the capillary tube is a mixture of air and saturated water vapour.

Trial Questions

1. A closed vessel contains a mixture of air and water vapour at 27°C at a total pressure of $1.07 \times 10^5 Nm^{-2}$. The water vapour is just saturated at this temperature. Calculate the total pressure in the vessel if:

(i) The temperature is raised to 60°C.

(ii) The temperature is lowered to 17⁰C (Saturated vapour pressure of water at 17⁰C is $1.9 \times 10^3\ Nm^{-2}$ and that at 27⁰C is $3.7 \times 10^5\ Nm^{-2}$)

[Ans: $1.188 \times 10^5\ Nm^{-2}, 1.018 \times 10^5\ Nm^{-2}$]

3. Moist air at 50°C and a pressure of 760mmHg is contained in a sealed vessel. When the vessel is cooled, condensation starts at 20°C. What will be the total pressure at 10°C? (Saturated

vapour pressure at 50°C, 20°C and 10°C are 92.0mmHg, 17.5mmHg and 9.0mmHg respectively) [Ans: 658mmHg]

4. In a Boyle's law experiment using damp air, the following results were obtained. Initial pressure (air saturated) = 8.5kPa, Pressure when volume is reduced to half the initial volume = 16kPa, Pressure when the volume is reduced to a third of the initial volume = 23kPa.
   (i) Show that the vapour exerts its saturation vapour pressure when the volume is reduced to half its initial value.
   (ii) Calculate the saturated vapour pressure at the temperature of the experiment.
   (iii) Calculate the initial pressure of the water vapour. [Ans: 2kPa, 1.5kPa]
5. A volume of 2000cc of oxygen at 15°C and pressure of 753mmHg has been collected over water. Find the volume of dry oxygen at s.t.p if the saturated vapour pressure of water at 15°C is 12.78mmHg. [Ans:$1.846 \times 10^3 cm^3$]
6. Two short threads of water confine a sample of air in a uniform capillary tube held horizontally. At a temperature of 20°C and atmospheric pressure of 76cmHg, the air bubble is 5cm long. When the tube is warmed to 60°C, the bubble is 6.9cm long. Find the saturated vapour pressure of water at 60°C if its value at 20°C is 18mmHg. Neglect surface tension effects. [Ans: 14.89 mmHg]

CHAPTER 19: THERMODYNAMICS

Thermodynamics is a science which deals with the relationship between heat and work done by a gas by considering thermal effects using quantities like pressure, temperature, volume and internal energy of a gas. The internal energy of an ideal gas is the kinetic energy of the thermal motion of its molecules. Its magnitude depends on the number of the molecules and the temperature.

LAW OF THERMODYNAMICS

Gas changes involve change in pressure, temperature, volume and heat quantity. These changes are governed by the law of thermodynamics. The law states that the total energy in a closed system is always a constant. That is, energy is conserved in any transfer of energy from one form to another. For example, when a gas is warmed, it expands and threw heat given to it (ΔQ) appears partly as an increase in its internal energy (ΔU) and partly as the energy required for the external work done (ΔW). Mathematically the law can be expressed as

$$\Delta Q = \Delta U + \Delta W$$

Where ΔQ is the heat supplied to the gas, ΔU is the internal energy gained by the gas, and ΔW is the work done on or by the gas. If P is the pressure applied to the gas and ΔV is the change in volume (expansion) of the, external work done the gas is given by

$$\Delta W = P.\Delta V$$

MOLAR HEAT CAPACITY

C_P is the molar capacity of a gas at constant at constant pressure. It's defined as the quantity of heat required to raise the temperature of one mole of a gas by 1°C or IK at constant pressure.

That is,

$$\Delta Q_P = nC_P\Delta T \text{ where } n = 1 \text{ mole and } \Delta T = IK$$

C_v is the molar capacity of a gas at constant at constant volume. It's defined as the quantity of heat required to raise the temperature of a gas by 1°C or IK, at constant pressure. That is,

$$\Delta Q_V = nC_v\,\Delta T \text{ where } n = 1\text{mole and } \Delta T = 1K$$

Relationship between C_P and C_v

Consider one mole (n = 1) of a gas warmed through IK ($\Delta T = 1$) at constant volume ($\Delta V = 0$). The heat required is given by

$$nC_v\Delta T = \Delta U + P.\,\Delta V$$

But $\Delta V = 0, n = 1\ mole\ and\ \Delta T = 1K.\ Therefore\ \Delta U = C_v$ ......(i)

Consider also one mole (n = 1) of a gas warmed through IK *(*ΔT = 1) at constant pressure. The heat required is given by;

$\Delta Q_P = \Delta U + \Delta W$

$nC_P\Delta T = \Delta U + P.\Delta V$

But $n = 1$ *mole and* $\Delta T = IK$. Therefore if V_1 *and* V_2 *are* the initial and final volumes of the gas, then $\Delta U = [C_P - P(V_2 - V_1)] = [C_P - (PV_2 - PV_1)]$

From gas laws, $PV_1 = RT$ *and* $PV_2 = R(T + \Delta T)$.

Therefore, $\Delta U = C_P - [R(T + \Delta T) - RT]$

Since ΔT = 1, then $\Delta U = C_P - R\Delta T$ ....(ii)

Equating the two equations *(i) and(ii)* gives;

$C_v = C_P - R$

$C_P - C_v = R$

This is the Relationship between C_P and C_v.

Why C_P is greater than C_v

At constant pressure, the heat energy supplied is partly taken in to increase internal energy and also used to do external work to increase the volume of the gas. However at constant volume, there's no external work done to increase the volume of the gas. Therefore, heat supplied only increases internal energy of the gas. Thus C_P is greater than C_v.

REVERSIBLE AND IRREVERSIBLE PROCESSES

A reversible process is one which can be made to go in the reverse direction through an infinitesimal change in the conditions causing it.

Examples of a reversible process include:

- Melting of ice.
- Expansion of a gas by a frictionless piston.

Irreversible processes

An irreversible process is one which can't be made to go in the reverse direction under the same conditions causing it.

Examples of irreversible processes include:

- Burning of kerosene.
- Expansion of a gas by a piston against friction.

ISOTHERMAL PROCESS

Isothermal process is a process that takes place at constant temperatures.

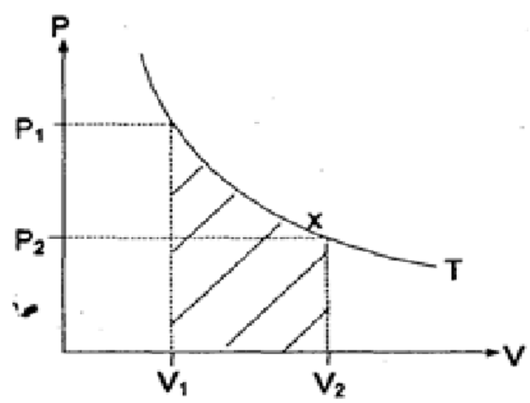
Isothermal change is a change the change in pressure and volume of fixed mass of a gas at constant temperature.

Reversible isothermal change is a change taking place at a constant temperature and can be taken from the final to initial states through an infinitesimal change in the values of temperature and pressure.

Conditions for achieving an isothermal process

- The gas container must be surrounded by a constant temperature bath to prevent temperature changes.
- The gas container must be thin walled to allow heat transmission between the gas and the constant temperature bath.
- The process must be carried out slowly to allow time for heat transmission.
- When carrying out the process, a light frictionless piston is used so that energy is on wasted in overcoming the weight of the piston and the frictional force.

Work done by a gas during isothermal change



Consider n moles of a gas undergoing a reversible isothermal expansion at a temperature T such that its volume changes from V_1 to V_2. The work done is given by;

$$\Delta W = P.\Delta V$$

From $PV = nRT$, it implies that $P = \left(\frac{nRT}{V}\right)$

Thus $\Delta W = \left(\frac{nRT}{V}\right)\Delta V$

$$\int_0^w dw = \int_{v_1}^{v_2}\left(\frac{nRT}{V}\right)dV$$

$$W = nRT[\ln V]_{v_1}^{V_2}$$

$$W = nRT\ln\left(\frac{V_1}{V_2}\right)$$

This is the expression for work done by the gas during isothermal change or process. If W is positive, work is done by the gas and thus, isothermal expansion occurs. . If W is negative, work is done on the gas and thus, isothermal contraction occurs.

ADIABATIC PROCESSES

Adiabatic process is a process which takes place in such a way that no heat enters or leaves the gas.

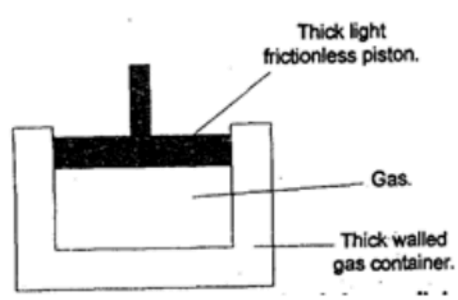
Adiabatic change is a change in temperature, pressure and volume of a fixed mass of a gas when no heat is allowed to enter or leave the gas. (i.e $\Delta Q = 0$).

Examples of adiabatic changes include:

- Sound transmission in air.
- A bursting tyre or a ball.
- Inflating a tyre or a ball,
- The compression stroke or power stroke of a four stroke cycle engine.

Conditions for achieving an adiabatic process

- The gas container must be thick walled and poorly conductive.
- The process must be carried out rapidly to avoid heat transmission exchanges,
- When carrying out an adiabatic process, the piston used must be light, frictionless and thick.



Why sound transmission in air is an adiabatic process

When sound passes through air, it makes a wave consisting of compression and rarefactions at a frequency of about 500Hz and of very short amplitude. This process of compression and rarefaction is fast enough not to allow significant transfer of energy from the system.

The air through which the sound moves is so vast that it's considered to be a well-insulated system. Also, the compressions and rarefactions cause changes in pressure, temperature and volume. Thus, transmission of sound in air is an adiabatic process.

Why adiabatic expansion of a gas results into cooling

In an adiabatic process heat, heat is not allowed to enter or leave the cylinder containing the gas. When a gas expands adiabatically, its molecules bounce off the moving piston with reduced speeds thus reducing the mean kinetic energy of the molecules. Since mean kinetic energy is proportional to absolute temperature, and heat is not allowed to enter the cylinder containing the gas, the temperature of the gas will therefore decrease. This results into cooling.

Equations for an adiabatic process

1. The expression relating pressure and volume for an adiabatic process is;

$PV^{\gamma} = constant$

2. For the equation relating pressure and temperature we substitute $V = \frac{RT}{P}$ in the above equation.

That is, $P\left(\frac{RT}{P}\right)^{\gamma} = constant$

$$P^{(1-\gamma)}T^{\gamma} = \frac{constant}{R^{\gamma}}$$

Since R is also a constant, then $\frac{constant}{R^{\gamma}}$ also a constant. Thus

$$P^{(1-\gamma)}T^{\gamma} = constant$$

3. For the equation relating volume and temperature we substitute for $P = \frac{RT}{V}$. That is,

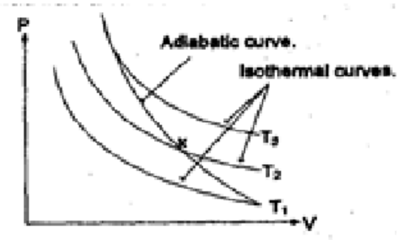
$$\left(\frac{RT}{V}\right)V^{\gamma} = C$$

$$V^{(\gamma-1)}T = \frac{C}{R}$$

Since R is also a constant, then $\frac{C}{R}$ is also a constant. Thus;

$$V^{(\gamma-1)}T = constant$$

Adiabatic and isothermal curves



Why at any point the gradient of an adiabatic curve is steeper than an isothermal curve

For isothermal process

$$PV = constant = C$$

$$\frac{d}{dV}(PV) = \frac{d}{dV}(C)$$

$$P + V\frac{dP}{dV} = 0$$

$$\frac{dP}{dV} = -\frac{P}{V}$$

This is the gradient for an isothermal curve.

For an adiabatic process,

$$PV^{\gamma} = constant = C$$

$$\frac{d}{dV}(PV^{\gamma}) = \frac{d}{dV}(C)$$

$$P(\gamma V^{\gamma-1}) + V^{\gamma}\frac{dP}{dV} = 0$$

$$\frac{\gamma PV^{\gamma}}{V} = -V^{\gamma}\frac{dP}{dV}$$

$$\frac{dP}{dV} = -\gamma\frac{P}{V}$$

This is the gradient of the adiabatic process.

Therefore, the gradient of an adiabatic expansion is $\boldsymbol{\gamma}$ times that of isothermal. The negative sign is because pressure decreases as volume increases.

Work done by an adiabatic process

Consider n moles of a gas undergoing a reversible adiabatic expansion from a volume a V_1 to V_2 such that its temperature changes from T_1 to T_2. The work done is given by

$$\Delta W = \Delta Q - \Delta U$$

Since no heat is allowed to enter or leave the gas, $\Delta Q = 0$. Therefore,

$$\Delta W = 0 - (nC_V\Delta T)$$

$$\int_0^W dw = -\int_{T_1}^{T_2}(nC_V)dT$$

$$W = -nC_V(T_2 - T_1)$$

Also from $PV = nRT$, it implies that $T = \left(\frac{PV}{nR}\right)$

Thus $W = -\,nC_V\left[\left(\frac{PV_2}{nR}\right) - \left(\frac{PV_1}{nR}\right)\right]$

$$W\left(\frac{R}{C_V}\right) = -(PV_2 - PV_1)$$

But $C_P - C_V = R$

Therefore $W\left(\frac{C_P - C_V}{C_V}\right) = P(V_1 - V_2)$

$$W\left(\frac{C_P}{C_V} - 1\right) = P(V_1 - V_2)$$

Also, $\frac{C_P}{C_V} = \gamma$.

Thus $W(\gamma - 1) = P(V_1 - V_2)$

$$W = \frac{P}{\gamma - 1}(V_1 - V_2)$$

This is the expression for work done by an adiabatic process.

NOTE:

1. For a mono-atomic gas [e.g. helium], $\gamma = 1.67$.
2. For a di-atomic gas [e.g. oxygen *(0_2)*, nitrogen *(N_2)* hydrogen (H_2)], γ =1.40.
3. For a poly-atomic gas [carbon dioxide ($C0_2$)] $\gamma = 1.30$

Other processes

1. Isovolumetric process is a process which takes place when a gas is heated or cooled at constant volume.
2. Isobaric process is a process which takes place when a gas is heated or cooled at constant pressure.

WORKED EXAMPLES

1. Calculate the work done against atmospheric pressure when 5kg of water turns into vapor of 1.01 x 10^5Pa. (Density of water = *1000kgm^{-3}*, density of vapor = 0.598kgm^{-3})

Solution

$$volume\ of\ water, V_1 = \frac{mass}{density} = \frac{5}{1000} = 0.005m^3$$

$$volume\ of vapour, V_2 = \frac{mass}{density} = \frac{5}{0.598} = 8.361m^3$$

Work done against atmospheric pressure

$$\Delta W = P.\Delta V$$

$$\Delta W = 1.01 \times 10^5 \times (8.361 - 0.005) = 8.44 \times 10^5 J$$

2. A vessel contains 1.5 x 10 $^{-3}$m^3 of an ideal gas at a pressure of $8.7 \times 10^4 Nm^{-2}$ and a temperature of 25°C. The gas is compressed isothermally and reversibly to a volume of 9 x 10^{-4}m^3. Calculate the work done in compression of a gas.

Solution

$$V_1 = 1.5 \times 10^{-3} m^3, V_1 = 9 \times 10^{-4} m^3$$

$$T = 25 + 273 = 298K, P_1 = 8.7 \times 10^4 Nm^{-2}$$

$$\Delta W = P_1 V_1 \ln\left(\frac{V_2}{V_1}\right)$$

$$\Delta W = 8.7 \times 10^4 \times 1.5 \times 10^{-3} \ln\left(\frac{9\times10^{-4}}{1.5\times10^{-3}}\right) = -66.67J$$

3. Nitrogen gas is trapped in the container by a movable piston. If the temperature of the gas is raised from 0°C to 50°C at a constant pressure of 4 x 10^{5}Pa and the total heat added is 3 x 10^{4}J. Calculate the work done by the gas. (C_P of nitrogen = 29.1 Jmol^{-1}K^{-1}, γ = 1.4)

Solution

$$\Delta Q = 3 \times 10^4 J, T_1 = 0^0C, T_2 = 50^0C$$

From $\Delta Q = nC_P\Delta T$

Number of moles of nitrogen gas

$$n = \frac{\Delta Q}{C_P\Delta T} = \frac{3\times10^4}{29.1\times(50-0)} = 20.62\ moles$$

From, $\gamma = \frac{C_P}{C_V}$

$C_V = \frac{C_P}{\gamma} = \frac{29.1}{1.4} = 20.79\ Jmol^{-1}K^{-1}$

Internal energy, $\Delta U = nC_V\Delta T$

$\Delta U = 20.62 \times 20.79 \times (50 - 0) = 2.143 \times 10^4 J$

From $\Delta Q = \Delta U + \Delta W$

Work done by a gas, $\Delta W = \Delta Q - \Delta U$

$\Delta W = 3 \times 10^4 - 2.143 \times 10^4 = 8.566 \times 10^3 J$

4. The temperature of one mole of helium gas, at a pressure of 1 x 10^5Pa, increases from 20°C to 100°C when the gas is compressed adiabatically. Find the final pressure of the gas. Take γ =1.67.

Solution

$T_1 = 20 + 273 = 293K, T_2 = 100 + 273 = 373K, P_1 = 1 \times 10^5 Pa, P_2 =?$

$\frac{P_1{}^{\gamma-1}}{T_1{}^{\gamma}} = \frac{P_2{}^{\gamma-1}}{T_2{}^{\gamma}}$

$P_2{}^{\gamma-1} = \left(\frac{T_2}{T_1}\right)^{\gamma} P_1{}^{\gamma-1}$

$P_2{}^{0.67} = \left(\frac{373}{293}\right)^{1.67} \times (1 \times 10^5)^{0.67} = 3.35 \times 10^3$

$P_2 = \sqrt[0.67]{3.35 \times 10^3} = 1.825 \times 10^5 Pa$

5. A vessel containing 1.5 x 10^{-3}m^3 of an ideal gas at a pressure of 8.7×10^{-2} Pa and a temperature of 25°C is compressed isothermally to half its original volume and then allowed to expand adiabatically to its original volume.

(a) Calculate the final temperature and pressure of the gas. Take γ = 1.41.

(b) Sketch a P-V graph for the whole process.

(c) Calculate the work done during the isothermal process.

Solution

Initially $P_1 = 8.7 \times 10^{-2}$ Pa, $V_1 = 1.5 \times 10^{-3} m^3$, $T_1 = 25 + 273 = 298K$

After isothermal compression

$V_2 = 0.5 \times 1.5 \times 10^{-3} = 7.5 \times 10^{-4} m^3, T_2 = 298K, P_2 =?$

$P_1V_1 = P_2V_2$

$P_2 = \frac{P_1V_1}{V_2} = \frac{8.7\times10^{-2}\times1.5\times10^{-3}}{7.5\times10^{-4}} = 0.174\ Pa$

After adiabatic expansion

$V_3 = 1.5 \times 10^{-3} m^3$, $T_3 = ?$, $P_3 = ?$, $\gamma = 1.41$

$$P_2 V_2^{\gamma} = P_3 V_3^{\gamma}$$

$$P_3 = \left(\frac{V_2}{V_3}\right)^{\gamma} \times P_2$$

$$P_3 = \left(\frac{7.5\times10^{-4}}{1.5\times10^{-3}}\right)^{1.41} \times 0.174 = 6.55 \times 10^{-2} Pa$$

Final pressure is $6.55 \times 10^{-2} Pa$

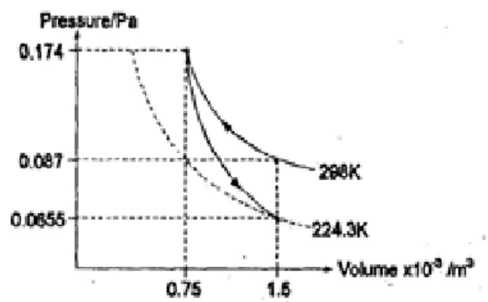
$$T_2 V_2^{\gamma-1} = T_3 V_3^{\gamma-1}$$

$$T_3 = \left(\frac{V_2}{V_3}\right)^{\gamma-1} \times T_2$$

$$T_3 = \left(\frac{7.5\times10^{-4}}{1.5\times10^{-3}}\right)^{0.41} \times 298 = 224.3K$$

Final temperature is 224.3K

(b)



(c) Work done during isothermal process

$$\Delta W = P_1 V_1 \ln\left(\frac{V_2}{V_1}\right)$$

$$\Delta W = 8.7 \times 10^{-2} \times 1.5 \times 10^{-3} \ln\left(\frac{7.5\times10^{-4}}{1.5\times10^{-3}}\right) = -9.046 \times 10^{-5} J$$

Trial Questions

1. A mass of air initially occupying a volume of 2000cc at a pressure of 760mmHg and temperature 20°C is expanded adiabatically and reversibly to twice its volume. It's then reduced adiabatically to a volume of 3000cc.

(i) Sketch a P-V graph to show the processes.

(ii) Calculate the final temperature and pressure of the air. Take $\gamma = 1.4$

[Ans: 249.2K, 430.8mmHg]

2. A gas initially occupying a volume of one litre at 273K and 1.01 x 10^5Pa is compressed isothermally to half its original volume. It's then allowed to expand adiabatically to the original v/olume.

(i) Calculate the final temperature and pressure of the gas.

(ii)Indicate the process on a P-V diagram. [Ans: 206.9K]

3. An ideal gas at a pressure of 2 x 10^6Pa occupies a volume of 2 x $10^{-3}m^3$ at 47.5°C. The gas expands adiabatically to a final pressure of 1.1 x 10^7Pa. Calculate the:

(i) Number of moles of the gas.

(ii) Final volume of the gas.

(iii) Final temperature of the gas. Take $\gamma = 1.4$.

[Ans: 1.502moles , $5.92 \times 10^{-4} m^3$, 521.6K]

4. An ideal gas of volume 100cc at s.t.p expands adiabatically until its pressure drops to a quarter its original value. Find the new volume and temperature if the ratio of the specific heat capacities is 1.4. [Ans: 269.2cc, 183.7K]
5. An ideal gas of volume one litre at s.t.p expands at constant pressure to a volume of three litres. Calculate:
   (i) The work done by the gas.
   (ii) The final temperature of the gas. [Ans: $2.02\times 10^5 J$, $819K$]

CHAPTER 20: NUCLEAR PHYSICS

TYPES OF RADIATIONS AND THEIR PROPERTIES

There are three types of radiations namely: alpha particles, beta particles and gamma rays.

By definition;

- Alpha particles are helium atoms which have lost two electrons. The symbol for an alpha particle is ${}^{4}_{2}He$.
- Beta particles are high energy electrons .the symbol for a beta particle is ${}^{0}_{-1}e$.
- Gamma rays are electromagnetic waves of the shortest wavelength.

Properties of alpha particles

- They are positively charged with a relative charge of +2.
- They have a relative mass of 4.
- They have a velocity of about a tenth $\left(\frac{1}{10}\right)$ to that of light.
- They have the least penetrating power; that is, they can penetrate 70mm of air and a fraction in millimeters in aluminum.

Properties of beta particles

- They are negatively charged with a charge of -1.
- They have negligible mass.
- They are identical to electrons emitted in cathode rays,
- They have higher penetrating power than alpha particles; that is, they can penetrate 7mm of aluminum and about 3mm of lead.

Properties of gamma rays

- These are not particles but electromagnetic rays with short wave length, shorter than x-rays.
- They have the velocity of light.
- They are extremely penetrating; that is, they can penetrate about 70mm of lead.
- They are not affected by electric and magnetic field. This is because they have no charge.
- They are dangerous to living creatures. This is because they are highly penetrative.

Similarities between alpha and beta particles

- Both have charges
- Both cause ionization of gases
- Both are deflected by gases

Similarity among alpha particles, beta particles, and gamma rays

- all cause ionization of gases
- all have penetrating power

Radioisotopes and their uses

Radioisotopes are nuclides which are unstable and undergo radioactive decay emitting alpha particles, beta particles, and gamma rays during which they return to stable form.

Some uses of radioisotopes

- They are used during the assessment of the volume and concentration of blood in patients.
- They are used to determine the rate of wear of piston rings of an engine and that of car tyres.
- They are used to detect leaks in underground pipelines carrying water, oil e.t.c
- They are also used in food processing industries for sterilization. For example meat can be made to stay fresh for over 15 days using gamma rays.
- They are used in paper industries to check the thickness of paper. The thickness of a paper can be checked by a beta source below the paper and a Geiger- Muller tube and a counter above it.
- They are also used in radiography in the treatment of cancer.

Gamma rays are used in radiography where they are replacing x-rays in the treatment of cancer.

- They are also used in carbon-14 dating

CARBON-14 DATING

Carbon-l4 dating is used to determine the age of a dead plant or tree (or fossils)

How carbon-14 is used to determine the age of a dead plant

Carbon-14 forms radioactive carbon dioxide which may be taken in by living plants during the process of photosynthesis.

When the plant dies, no fresh carbon is taken in and carbon-14 in the dead plant starts to decay by emission of beta particles. Using a Geiger-Muller tube, the activity *(A)* of the dead plant is determined. Also, the activity *(A_0)* of a similar plant which is still living is measured.

Since the half-life $\left(t_{\frac{1}{2}}\right)$ of carbon-14 is known, the time *(t)* since the plant died can be estimated from the formula $A = A_0e^{-\lambda t}$ where $\lambda = \frac{0.693}{t_{\frac{1}{2}}}$

Qn: Explain one industrial and one biological use of radioisotopes.

Solution

- Radioisotopes are used as tracers in industries. For investigation of the flow of a liquid in an underground pipe of sewage or water, a little radioactive solution is added to the liquid being pumped. A temporarily high activity around the leak will be detected from the ground above.
- They are also used in the treatment of cancer. Minute radioactive particles are attached to the antibodies before they are given to the patient and the radioactivity reveals the cancer cells. Further tiny doses of radioactivity fixed to the antibodies can then destroy the harmful cancer cells with minimum damage to the surrounding healthy cells.

RADIATION HAZARDS AND PRECAUTIONS

The radiation hazards to human beings arise from;

- Exposure of the body to external radiation and,
- Inhalation or ingestion of radioactive substances.

Alpha particles are stopped by the outer layer of the skin since their penetrating power is small. However, if they penetrate into the body, they are dangerous as they damage some body organs. Therefore, alpha particles can damage skin tissues and eye sight.

Beta particles and gamma rays are more penetrating than alpha particles and destroy cells in body tissues. When the body is exposed to them, they upset the natural chemical reactions. This may lead to injury or death. They can also cause radiation burns (redness and sores on the skin) and delayed effects such as cancer.

Health hazards of radioactive substances

- They can cause damage to body cells due to creation of ions which upset or destroy them,
- Radiation can also cause immediate damage to tissue and, according to the dose, is accompanied by radiation burns (that is, redness of the skin followed by blistering and sores which are slow to heal)
- They can cause death when one is exposed to too much radiation,
- Hereditary defects (deformation of offspring at birth) may occur in succeeding generations due to genetic damage,
- Delayed effects such as cancer, leukemia and eye cataracts may also appear many years later.

General safety precautions taken when handling radioactive sources

- Don't direct the sources to a person or to yourself,
- Don't drink, eat, or smoke when using radioactive sources,
- Wounds and cuts should be covered,
- Use forceps to handle sources of radiation,
- Thoroughly wash your hands after using radioactive sources,
- When the sources aren't in use, they should be kept in lead boxes (lead is a heavy metal and they can't penetrate it).

Safety precautions when using radioactive sources in industries

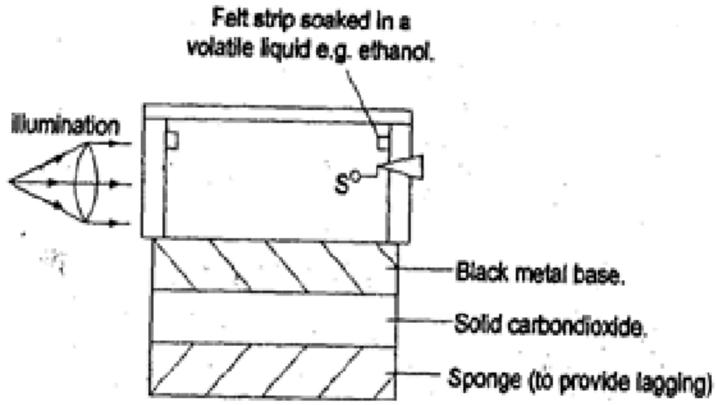
- Workers should wear special budges containing photographic films,
- Worker should also wear special clothing protected by lead.
- Tweezers and remote reading machines should be used.
- Waste products of nuclear power stations should be properly disposed. For example, they can be buried underground so that they don't cause harm.

NUCLEAR RADIATION DETECTORS

In a nuclear radiation detector, energy is transferred from the radiation to atoms of the detector and may cause ionization of a gas in an ionization chamber, a Geiger-Muller tube, and a cloud chamber. Nuclear radiation detectors including:

a) Diffusion cloud chamber.
b) Expansion/Wilson cloud chamber.
c) Ionization cloud chamber.
d) Geiger-Muller tube.

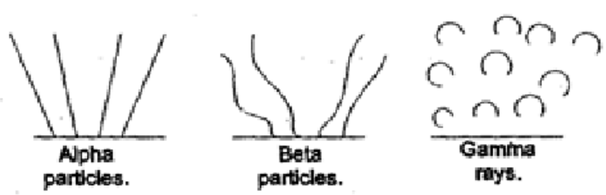
DIFFUSION CLOUD CHAMBER



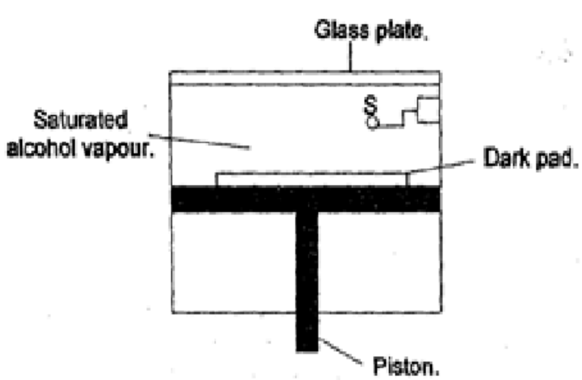
S is the source of radiation.

The top part of the chamber is at room temperature while the bottom part of the chamber is maintained at a very low temperature by the dry/solid carbon dioxide and this creates a temperature gradient between the top and the bottom. Ethanol, which is a volatile liquid, vaporizes from the felt strip soaked in ethanol in the warm top part of the chamber and diffuses downwards. At some distance above the metal base, there's a region of super saturation of ethanol vapour.

When the shield on S is removed, condensation of the vapour on the ions formed by *S* occurs. The paths of the ionizing radiations are traced by a series of small drops of condensation. The thickness and length of the path indicate the extent to which ionization has taken place. These tracks are observed against a black metal base by strong illumination. Shape of the tracks is as shown below.



EXPANSION/WILSON CLOUD CHAMBER



S is the source of radiation.

Alcohol is placed on a dark pad on a piston. When the piston is moved down quickly, the air in the chamber undergoes an adiabatic expansion and cools. The dust nuclei are carried away after a few expansions by drops forming on them after condensation.

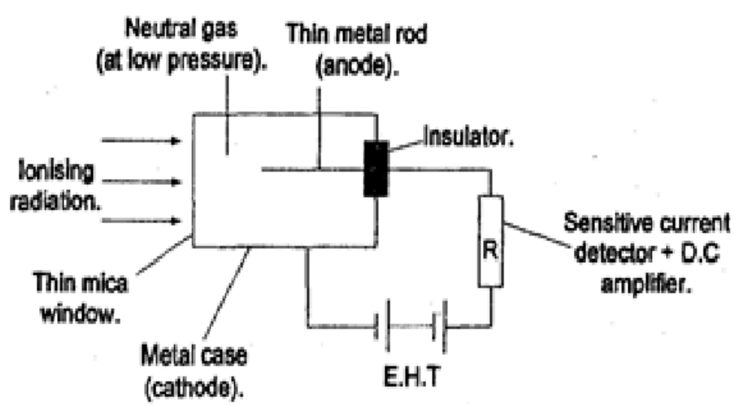
The dust free air is now subjected to a controlled expansion making the air super saturated. The air is then exposed to an ionizing radiation from S. The water droplets will collect around the ions produced and these water droplet reflect light when illuminated thus enabling the detection of the paths of the ionizing radiation.

The thickness and length of the paths indicate the extent to which ionization has taken place.

Uses of a Wilson cloud chamber

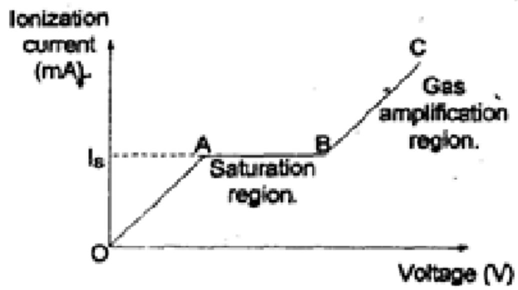
- It's used for the study of radioactive isotopes.
- It's used for recording the tracks of ionization particles.
- It's also used to determine energy of radioactive particles.
- By counting drops in cloud tracks, the specific ionization can be determined.

IONIZATION CHAMBER



When an ionizing radiation enters the chamber, ion pairs are produced. The positively charged ions move to the cathode while the negatively charged ions (electrons) move to the anode. Current flows through the external circuit and the voltage pulses are amplified and recorded. The Extra High Tension (EHT) is set to a value such that a constant current, I_s flows. In this setting the energy of the incoming radiation is proportional to current, I_s. Alpha particles produce a current of order $1 \times 10^{-10} A$ beta particles and gamma rays produce (or cause) a much smaller current because they cause less ionization.

A graph of ionization current against voltage



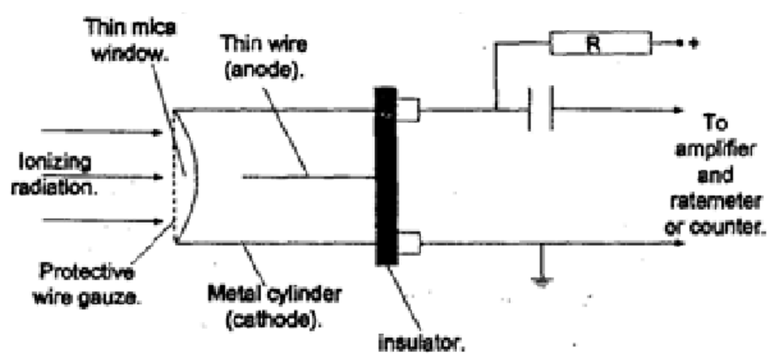
Along OA, ionization current is proportional to voltage. This is because from O to A, voltage is low thus some positive ions and electrons produced can recombine to form neutral atoms.

Along AB, ionization current is independent of the changes in the voltage. This is because all the ion pairs produced reach the electrode thus resulting into a constant current called saturation current.

Along BC, the electrons produced have sufficient energy to cause ionization of other gas atoms. This results in rapid multiplication of ions in the chamber. This is called secondary ionization.

NOTE: The ionization chamber is operated in **region AB** and this is because ionization current is independent of the changes in voltage along AB.

GEIGER-MULLERTUBE

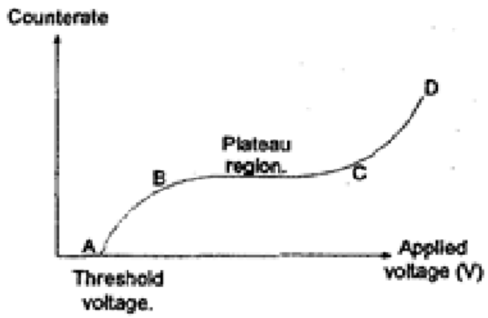


The Geiger-Muller tube is connected to the counter or rate meter and a source of radiation is then brought near a mica window. When this radiation enters the chamber through the mica window, ion pairs are produced.

Electrons move towards the anode at a very high speed because of the high potential difference between the anode and the cathode. This makes electrons to move with a high kinetic energy and thus capable of ionizing several other atoms. This leads to ion amplification.

When the electrons reach the anode, a discharge occurs causing a pulse current through the resistor

(R). The voltage pulse which results is then amplified and registered by a rate meter or a scaler. The count rate is read from the rate meter which is equal to the number of the in-coming ionizing particles entering the chamber per second. A graph of count rate against applied voltage



Along AB, not all the ions reach the electrodes thus some recombine to form other neutral atoms. Along BC, all the ions reach the electrodes. The slight slope along BC is due to occasional failure of the quenching action and end effects.

Along CD, there's uncontrolled multiplication of ions.

NOTE:

i. The Geiger-Muller tube is operated in region BC and this is because all the particles in this region produce a sufficiently high voltage pulse for counting although their initial ionization may be different.

ii. A rate meter measures count rate (number of pulses per second)

iii. A sealer records brief currents called pulses or ionization current of the individual ionizing radiation.

Functions of some parts of a Geiger-Muller tube

Thin mica window:

It allows easy entry of ionizing particles into the Geiger-Muller tube.

Argon gas at low pressure:

When atoms collide with the neutral atoms, ion pairs are formed. When pressure is low, gas molecules are much further apart and ions move with minimum interference to the electrodes.

Halogen gas mixed with argon:

The mixture forms a quenching agent which prevents secondary ionization.

An anode in the form of a wire:

It increases the strength of the electric field intensity.

Importance of using inert gases in a Geiger-Muller tube

Inert gases, for example argon, are suitable filling gases since they don't easily attract electrons and ensure maximum sensitivity.

Advantage of a Geiger-Muller tube over an ionization chamber

Geiger-Muller tubes have excellent sensitivity in that they can detect even a single ion pair.

NB: Dead time is defined as the time taken for the positive ions to travel towards the cathode in a Geiger-Muller tube.

RADIOACTIVE DECAY

The changes accompanying the emission of radiation from radioactive substances differ from the ordinary chemical changes. They are spontaneous, uncontrollable, and are unaffected by chemical combination and physical conditions such as pressure and temperature. It involves the nucleus of an atom (not its extra electrons as in chemical changes) and is an attempt by an unstable nucleus to become more stable.

Note: The spontaneous disintegration of an unstable heavy nucleus by emission of a radiation (alpha particles, beta particles, or gamma rays) is known as **radioactivity**. Energy is always released in this case.

Emission of alpha particles

An alpha particle is a helium nucleus consisting of 2 protons and 2 neutrons. When an atom decays by emission of an alpha particle, its nucleon / mass number (A) decreases by 4, whereas its atomic / proton number (Z) decreases by 2. For example, when Uranium of mass number 238 ,and atomic number 92 emits an alpha, it decays to thorium of mass number 234 and atomic number. (It becomes the atom of an element two places lower in the periodic table).

$$^{238}_{92}U \longrightarrow {}^{238-4}_{92-2}Th + {}^{4}_{2}He$$

$$^{238}_{92}U \longrightarrow {}^{234}_{90}Th + {}^{4}_{2}He$$

Emission of beta particles

When beta decay occurs, a neutron changes into a proton and an electron. The proton remains in the nucleus and the electron is emitted as a beta particle. The new nucleus has the same mass number, but its proton number increases by one since it has one more proton. For example, active carbon, called carbon-14, which has mass number of 14 and proton number of 6 decays by beta emission to nitrogen having the same mass number(14) but proton number of 7. (It becomes the atom of an element one place higher in the periodic table).

$${}^{14}_{6}C \longrightarrow {}^{14}_{6+1}C + {}^{0}_{-1}e$$

$${}^{14}_{6}C \longrightarrow {}^{14}_{7}N + {}^{0}_{-1}e$$

Emission of gamma rays

The emission of gamma rays is explained by considering that the nuclei (as well as atoms) have energy levels and if an alpha or a beta particle is emitted, the nucleus is left in an excited state. A gamma ray photon is emitted when the nucleus returns to its ground state. The existence of different nuclear energy levels would account for the 'line-type' energy spectrum of gamma rays.

TERMS AND DEFINITIONS

Atomic or (proton number (Z): The atomic or proton number Z of an atom is the number of protons in its nucleus.

Mass or nucleon number *(A):* The mass or nucleon number A of an atom is the number of nucleons in the nucleus.

It follows that if N is the neutron number of the nucleus, that is, the number of nucleons it contains, then

$$A = Z + N$$

An atom X with atomic number Z and mass number A is represented by ${}^{A}_{Z}X$. For example, ${}^{10}_{5}B$ represents an atom of boron with; Atomic number = 5, Mass number = 10 and Neutron number = $(10 - 5) = 5$.

DECAY LAW

It states that the rate of disintegration is directly proportional to the number of active atoms present.

Let N = Number of unstable atoms present at time t.

N_0 =Original number of atoms present at $t = 0$.

λ = Decay constant.

From the decay law, $-\frac{dN}{dt} \propto N$

The negative sign implies that N decreases as t increases. The radioactive decay constant (λ)

is the constant of proportionality in this expression, giving

$$\frac{dN}{dt} = -\lambda N$$

Putting integrals on both sides gives,

$$\int_{N_0}^{N} \frac{dN}{N} = \int_0^t -\lambda\, dt$$

$$[\ln N]_{N_0}^{N} = [-\lambda\, t]_0^t$$

$$\ln N - \ln N_0 = -\lambda t - 0$$

$$\ln\left(\frac{N}{N_0}\right) = -\,\lambda t$$

$$\frac{N}{N_0} = e^{-\lambda t} \quad \text{since } \ln \quad = \log_e$$

$$N = N_0 e^{-\lambda t} \quad \ldots.(i)$$

NB: **Radioactive decay constant, λ** is defined as the fractional number of atoms which decay per second.

HALF LIFE OF A RADIOACTIVE SUBSTANCE

Half-life $\left(t_{1/2}\right)$is defined as the time taken for the number of active nuclei present in a source at a given time to fall to half its value.

When $t = t_{1/2}$, $N = \frac{1}{2}N_0$

Substituting for 't' and 'N' in equation (i) gives;

$$\left(\frac{1}{2}N_0\right) = N_0 e^{-\lambda t_{1/2}}$$

$$\left(\frac{1}{2}\right) = e^{-\lambda t_{1/2}}$$

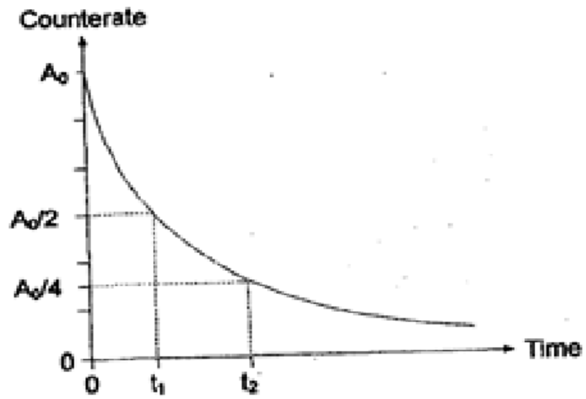
$$\ln 2 = -\lambda t_{1/2}$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

Determining half-life of a radioactive substance using a Geiger- Muller tube

The source of radiation is placed near a thin mica window of a Geiger-Muller tube.

The number of counts per second (activity), A is recorded by the counter at different values of time, t. The results are tabulated and a graph of count rate (activity) against time is plotted. The graph is as shown follows.



Time taken for A to fall to $\frac{1}{2}A_0 = (t_1 - 0)$

Time taken for $\frac{1}{2}A_0$ to fall to $\frac{1}{4}A_0 = (t_2 - t_1)$

Half life = Average time

$= \frac{1}{2}[(t_1 - 0) + (t_2 - t_1)]$

$= \frac{1}{2}t_2$

Where A_0 is the initial activity of the radio active source

ACTIVITY OF A RADIOACTIVE SOURCE

Activity (A) of a radioactive source is defined as the number of disintegrations of a radioactive source per second.

Let A = Activity of a radioactive source at time t.

A_0= Activity of a radioactive source at time t = 0.

$$A = \left|\frac{dN}{dt}\right|$$

From equation (i) , $N = N_0e^{-\lambda t}$

Therefore, $A = \left|\frac{d}{dt}\left(N_0e^{-\lambda t}\right)\right|$

$= \left|-\lambda N_0e^{-\lambda t}\right|$

$= \lambda N_0e^{-\lambda t}$

But $N = N_0e^{-\lambda t}$ thus $A = \lambda N \ldots\ldots(iii)$

This implies that if $A = \lambda N \quad \ldots\ldots(*)\ then\ A_0 = \lambda N_0 \ldots\ldots(**)$

Dividing equation (*) by (**) gives;

$$\frac{A}{A_0} = \frac{\lambda N}{\lambda N_0} = \frac{N_0e^{-\lambda t}}{N_0}$$

$$\frac{A}{A_0} = \frac{e^{-\lambda t}}{1}$$

$$A = A_0 e^{-\lambda t} \ldots\ldots\ldots (iv)$$

Steps taken to measure activity of a radioactive source using a Geiger-Muller tube

i. Measure the background count rate of the radioactive source, A_0.

ii. Measure the count rate of the source, A_1

iii. True activity of the source = $(A_1 - A_0)$.

UNIFIED ATOMIC MASS UNIT

This is a twelfth $\left(\frac{1}{12^{th}}\right)$ the mass of one atom of carbon-12 i.e. 1 mole of carbon-12 has 12g and 6.02×10^{23} atoms.

Mass of one atom of carbon-12 $= \frac{12}{6.02 \times 10^{23}} = 1.99 \times 10^{-23}$g

$$1U = \frac{1}{12} \times (mass\ of\ one\ atom\ of\ carbon - 12)$$

$$= 1.66 \times 10^{-24}\text{g}$$

But 1kg = 1000g

Therefore, $1U = \frac{1.66 \times 10^{-24}}{1000} = 1.66 \times 10^{-27}$kg

The electron volt

This is defined as the kinetic energy acquired by an electron when accelerated by a p.d of 1 volt

$$K.E = eV$$

$$K.E = 1.6 \times 10^{-19} \times 1 = 1.6 \times 10^{-19} J$$

$$\therefore\ 1eV = 1.6 \times 10^{-19} J$$

NUCLEAR FISSION

This is the disintegration of a heavy nucleus into two or more light nuclei, neutrons and emission of energy.

A fission reaction gives normally more energy than a fusion reaction e.g

$$^{235}_{92}U + ^{0}_{1}n \rightarrow ^{141}_{56}Ba + ^{92}_{36}Kr + 3^{0}_{1}n\ +\ energy$$

Such a reaction where one neutron produces more neutrons is called a chain reaction and can be used to produce atomic bombs.

NUCLEAR FUSION

This is the combining of two or more light nuclei to form a heavy nucleus with emission of energy e.g the fusion of two deuterium nuclei to produce helium.

$${}_{1}^{2}H + {}_{1}^{2}H \rightarrow {}_{2}^{3}He + {}_{0}^{1}n + Energy$$

High temperatures are required in nuclear fusion to provide the nuclei with the necessary energy to overcome their mutual electrostatic repulsion.

EINSTEIN'S RELATION OF MASS AND ENERGY

Einstein predicted that if the energy of a body changes by an amount *E,* its mass changes by an amount Δm given by the equation

$$E = \Delta mc^2$$

where *c* is the speed of light in a vacuum and has a value $3 \times 10^8 ms^{-1}$.

Using the above equation, unified mass can be expressed in terms of energy i.e.

$$\text{Energy of } 1U = \Delta mc^2 = (1.66 \times 10^{-27}) \times (3 \times 10^8)^2$$

$$= 1.494 \times 10^{-10} J$$

$$\text{But } 1eV = 1.6 \times 10^{-19} J$$

$$\text{Thus } 1U = \left[\frac{1.494 \times 10^{-10}}{1.6 \times 10^{-19}}\right] \approx 931 MeV$$

MASS DEFECT

This is the mass equivalent of energy required to split up the nucleus into its constituent nucleons.

The mass of the nucleus is always less than the total mass of neutrons and protons.

$$Mass\ defect = \begin{pmatrix} mass\ of\ protons \\ and\ neutrons \end{pmatrix} - \begin{pmatrix} mass\ of \\ the\ nucleus \end{pmatrix}$$

$$\text{Or } Mass\ defect = \begin{pmatrix} mass\ of\ nucleons \\ and\ electrons \end{pmatrix} - \begin{pmatrix} mass\ of\ the \\ atom/nucleus \end{pmatrix}$$

BINDING ENERGY

Binding energy is the energy released when nucleons come together to form a nucleus.

Binding energy can also be defined as the energy required to split the nucleus into individual nucleons.

Consider a helium nucleus:

$^4_2He = 4.0015U\ from\ the\ atomic\ mass\ scale; proton, p = 1.0073U, neutron, n = 1.0087U$

[Number of nucleons] = [Number of protons] + [Number of neutrons]

= 2 + 2 = 4

To form a helium nucleus, you need 2 protons and 2 neutrons

$$\begin{pmatrix} mass\ of\ the\ costituent \\ protons\ and\ neutrons \end{pmatrix} = 2p + 2n$$

$$= 2(1.0073U + 1.008U) = 4.0320U$$

Mass defect $\Delta m = 4.032U - 4.0015U = 0.0305U$

But $1U = 1.66 \times 10^{-27}$kg

Therefore $\Delta m = 0.0305 \times 1.66 \times 10^{-27} = 5.063 \times 10^{-29} kg$

Energy given out is the binding energy i.e

Binding energy = energy given out

$$= \Delta mc^2$$

$$= 5.063 \times 10^{-29} \times (3 \times 10^8)^2 = 4.557 \times 10^{-12} J$$

But $1eV = 1.6 \times 10^{-19} J$

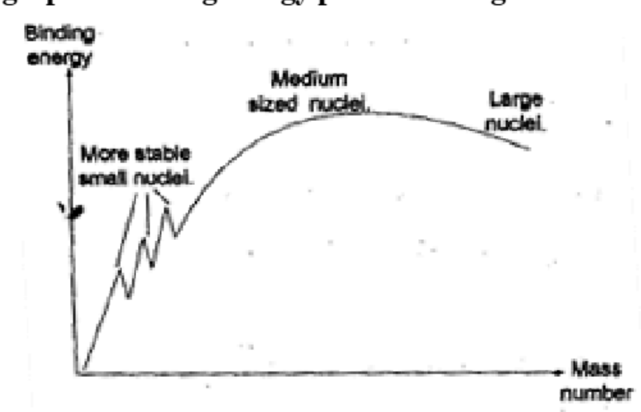
Therefore, binding energy $= \frac{4.557 \times 10^{-12}}{1.6 \times 10^{-19}} = 2.848 \times 10^7 eV = 28.48MeV$

Binding energy per nucleon = $\frac{total\ binding\ energy}{total\ number\ of\ nucleons}$

$$= \frac{28.48}{4} = 7.12MeV$$

Binding energy per nucleon is the ratio of the binding energy to the number of nucleons (protons and neutrons)

A graph of binding energy per nucleon against mass number



Very small nuclei are unstable because of the large surface energy. Small nuclei for which number of protons is equal to number of neutrons are more stable than those whose number of protons is not equal to their number of nucleons. This explains the presence of the spikes. Medium sized nuclei are the most stable because the nuclear forces are saturated i.e. each nucleon interacts strongly with a few neighbors. This explains the maxima of the curve.

Large nuclei are unstable because of electrostatic repulsion among the large number of protons. This explains the slight decrease at the end of the curve.

NB: Binding energy per nucleon is defined as the total energy required to split the nucleus of an atom into its constituent particles divided by the mass number of the atoms.

DETERMINING THE VOLUME OF BLOOD

A little of the patient's blood is removed and put in a solution containing a radioactive material of known half-life.

A carefully removed quantity of the mixture of the blood and the radioactive material is then reintroduced in the blood stream of a patient. After some time, the body will be evenly distributed with the radioactive material.

A small sample of blood is then removed and tested using a detector to determine its activity.

High concentration of the radioactive material in the blood (high activity) shows that the blood is not enough.

The volume of blood in the patient will be equal to the total activity expected after time t divided by the activity per unit volume of blood.

RATE OF WEAR OF A PISTON RING

The piston ring is radiated so that it can become radioactive. The mass of the piston ring and the initial activity of the piston ring are also determined using relevant instruments.

The ring is then installed in the engine and the engine is run continuously for a given time.

After this time, the activity of oil from the engine is determined using a detector.

$$\text{The mass worn off} = \frac{Activity\ of\ oil}{Activity\ expected\ after\ time\ t} \times total\ mass\ of\ the\ ring$$

$$\text{Rate of wear} = \frac{mass\ worn\ off}{time\ of\ use}$$

WORKED EXAMPLES

Where applicable, use the following constants

Electronic charge, $e = 1.6 \times 10^{-19}C$

Avogadro's number = $6.02 \times 10^{23} mol^{-1}$

Velocity of light in a vacuum, $c = 3 \times 10^{8}$

1. A radioactive source produces alpha particles each of energy 60MeV. If 20% of the alpha particles enter the ionization chamber and a current of 0.2μA flows. Find the activity of the alpha source if the energy needed to make an ion pair in the ionization chamber is 32eV.

Solution

$I = 0.2\mu\text{A} = 0.2 \times 10^{-6}A$

From $Q = It$, rate of flow of charge

$\left(\frac{Q}{t}\right) = I = 0.2 \times 10^{-6}Cs^{-1}$

From $Q = ne$, number of electrons emitted per second;

$$n = \frac{\left(\frac{Q}{t}\right)}{e} = \frac{0.2\times10^{-6}}{1.6\times10^{-19}} = 1.25 \times 10^{13}$$

Total energy needed to make all the ion pairs in the chamber per second

$$= \begin{pmatrix} number\ of\ electrons \\ emitted\ per\ second \end{pmatrix} \times \begin{pmatrix} energy\ required \\ to\ make\ an\ ion\ pair \end{pmatrix}$$

$$= 1.25 \times 10^{13} \times 32 = 4 \times 10^{14}eV = 4 \times 10^{8}MeV$$

Since each alpha particle requires *60MeV,* then, the number of alpha particles entering the chamber per second $= \frac{4\times10^{8}}{60} = 6.67 \times 10^{6}$

But only 20% of alpha particles enter the chamber per second. Let α be the number of alpha particles entering the chamber per second.

20% of $\alpha = 6.67 \times 10^{6}$

$$\alpha = \frac{6.67\times10^{6}}{0.2} = 3.33 \times 10^{7}$$

Therefore, the activity of the alpha source is 3.33×10^{7} s^{-1}

2. A mass of 4g of the nuclide $^{25}_{11}Na$ decays by emission of β-particles. Its half-life is 71s. Find the;

i. Number of $^{25}_{11}Na$ atoms initially present.

ii. Initial activity of the sample.

iii. Number of $^{25}_{11}Na$ atoms present after 20minutes.

Solution

$$M = 25gmol^{-1}, m = 4g, N_0 = ?, t_{1/2} = 71s$$

Number of moles, $n = \frac{N_0}{N_A} = \frac{m}{M}$

Therefore number of $^{25}_{11}Na$ atoms initially present ;

$$N_0 = \frac{m}{M} \times N_A = \frac{4}{25} \times 6.02 \times 10^{23} = 9.632 \times 10^{22}$$

(ii) $t_{1/2} = \frac{0.693}{\lambda} = 71$

$$\lambda = \frac{0.693}{71} = 9.761 \times 10^{-3} s^{-1}$$

Initial activity $A_0 = \lambda N_0 = 9.761 \times 10^{-3} \times 9.632 \times 10^{22}$

(iii)

$$\lambda t = 9.761 \times 10^{-3} \times 20 \times 60 = 11.71$$

Number of $^{25}_{11}Na$ atoms present after 20minutes;

$$N = N_0 e^{-\lambda t}$$

$$N = 9.632 \times 10^{22} \times e^{-11.71} = 7.909 \times 10^{7}$$

3. An isotope of Bismuth of mass number 200 has a half-life of $5.4 \times 10^3 s$. It emits alpha particles with energy of $8.2 \times 10^{-13} J$. Calculate for this isotope the;

i. Decay constant.

ii. Initial activity of 2×10^{-6} moles of the isotope.

iii. Initial power output of this quantity of the isotope.

Solution

$$t_{1/2} = \frac{0.693}{\lambda} = 5.4 \times 10^3$$

$$\lambda = \frac{0.693}{5.4 \times 10^3} = 1.283 \times 10^{-4} s^{-1}$$

(ii) Number of atoms, $N_0 = nN_A$

$$N_0 = 2 \times 10^{-6} \times 6.02 \times 10^{23} = 1.204 \times 10^{18}$$

Initial activity $A_0 = \lambda N_0$

$$A_0 = 1.283 \times 10^{-4} \times 1.204 \times 10^{18} = 1.545 \times 10^{14} s^{-1}$$

(iii) $\begin{pmatrix} initial \\ power\ output \end{pmatrix} = \begin{pmatrix} initial \\ activity \end{pmatrix} \times \begin{pmatrix} energy \\ released \end{pmatrix}$

$= 1.545 \times 10^{14} \times 8.2 \times 10^{-13} = 1.267 \times 10^{2} W$

4. Consider the following nuclear reaction.

$${}^{235}_{92}U + {}^{1}_{0}n \rightarrow {}^{133}_{x}A + {}^{y}_{40}R + 6{}^{1}_{0}n$$

i. Determine the values of x and y.

ii. What is the importance of this reaction?

Solution

(i) $235 + 1 = 133 + y + 6$

$y = 6$

$92 + 0 = x + 40 + 0$

$x = 52$

(ii) The reaction is for nuclear fusion. Therefore, energy is released during the process which can be changed to electricity at power stations and used in the manufacture of atomic bombs.

5 (a). The following is part of uranium-238 decay series

$${}^{238}_{92}U \quad \rightarrow^{(i)} \quad {}^{234}_{90}Th \ \rightarrow^{(ii)} \ {}^{234}_{91}Pa$$

Name the particles emitted at each of the stages (i) and (ii).

(b) Calculate the energy liberated when a helium nucleus ${}^{4}_{2}He$ is produced by fussing two deuterons nuclei ${}^{2}_{1}H$. (Give your answer in MeV)

[Mass of ${}^{4}_{2}He$= 4.004U, Mass of ${}^{2}_{1}H$ = 2.015U, 1U = 1.66 x 10^{-27}kg]

Solution

(a). For stage (i), let the particle be *x*.

$mass\ number\ = 238 - 234 = 4$

$atomic\ number = 92 - 90 = 2$

$particle\ is\ in\ the\ form\ {}^{4}_{2}x\ which\ is\ an\ alpha\ particle.$

For stage (ii), let the particle be *y*.

$mass\ number\ = 234 - 234 = 0$

$atomic\ number\ = 90 - 91 = -1$

$particle\ is\ in\ the\ form\ {}^{0}_{-1}y,\ which\ is\ a\ beta\ particle.$

(b)

$${}^{2}_{1}H + {}^{2}_{1}H = {}^{4}_{2}He + E$$

$2.015 + 2.015 = 4.004 + \Delta m$

$\Delta m = 4.03 - 4.004 = 0.026U$

$\Delta m = 0.026 \times 1.66 \times 10^{-27} = 4.316 \times 10^{-29} kg$

$E = \Delta mc^2$

$E = 4.316 \times 10^{-29} \times (3 \times 10^8)^2 = 3.884 \times 10^{-12} J$

$E = \frac{3.884 \times 10^{-12}}{1.6 \times 10^{-19}} = 2.428 \times 10^7 eV$

$E = 2.428 \times 10^7 \times 10^{-6} MeV = 24.28\ MeV$

6. The radioisotope ^{60}Co decays to ^{60}Ni by emission of a β- particle and two γ-photons. The half-life of ^{60}Co is 5.27 years.

(i) Calculate the maximum energy (in MeV) of the gamma radiation given off per disintegration.

(ii) Find the power of radiation emitted by 5g of ^{60}Co. [$Mass\ of$ ^{60}Co = 59.9338U, Mass of ^{60}Ni = 59.9308 U, mass of $_{-1}^{0}e = 0.0005U, 1U = 1.66 \times 10^{-27} kg$]

Solution

$^{60}Co \rightarrow {}^{60}Ni + {}_{-1}^{0}e + E$

$59.9338 = 59.9308 + 0.0005 + \Delta m$

$\Delta m = 0.0025 \times 1.66 \times 10^{-27} = 4.15 \times 10^{-30} kg$

$E = \Delta mc^2$

$E = 4.15 \times 10^{-30} \times (3 \times 10^8)^2 = 3.735 \times 10^{-13} J$

$E = \frac{3.735 \times 10^{-13}}{1.6 \times 10^{-19}} = 2.334 \times 10^6 eV = 2.334 MeV$

7. A steel piston ring contains 15g of radioactive iron $_{26}^{54}Fe$. The activity of $_{26}^{54}Fe$ is 3.7×10^5 disintegrations per second. After 100 days of continuous use, the crank case oil was found to have a total activity of 1.23×10^3 disintegrations per second. Find the

(i) Half-life of $_{26}^{54}Fe$

(ii) Average mass of iron worn off the ring per day assuming that all the metal removed from the ring accumulates in the oil.

Solution

54 g of $_{26}^{54}Fe$ contains $6.02 \times 10^{23} atoms$

15 g of iron contains $\frac{6.02 \times 10^{23}}{54} \times 15 = 1.67 \times 10^{23} atoms$

But $A = \lambda N$

$$3.7 \times 10^5 = \frac{0.693 \times 1.67 \times 10^{23}}{t_{1/2}}$$

$$t_{1/2} = 3.625 \times 10^{12}\ days$$

Activity expected after time t is given by; $A = A_0 e^{-\lambda t}$

$$A = 3.7 \times 10^5 e^{-(1.912 \times 10^{-11})} = 3.7 \times 10^5 \text{dis/sec}$$

$$\text{mass worn off} = \frac{Activity\ of\ oil}{Activity\ expected\ after\ time\ t} \times total\ mass\ of\ the\ ring$$

$$= \frac{1.23 \times 10^3}{3.7 \times 10^5} \times 15 = 0.04986\text{g} \approx 0.05\text{g}$$

$$\text{Average mass worn off per day} = \frac{mass\ worn\ off}{time\ of\ use}$$

$$= \frac{0.05}{100} = 5 \times 10^{-4}\text{g/day}$$

8. A small volume of a solution which contains a radioactive isotope of sodium had an activity of 1200 disintegrations per minute when it was first introduced in the blood stream of the patient. After 30 minutes, the activity of 1cm^3 of the blood was 0.5 disintegrations per minute. If the half-life of sodium is 15 minutes. Estimate the volume of blood in the patient.

Solution

$$A_0 = 1200\ dis\ /\ min, t_{1/2} = 15\ min$$

1cm^3 of blood has 0.5 dis/min after 30 min

$$\lambda = \frac{0.693}{15} = 0.0462\ min^{-1}$$

$$A = A_0 e^{-\lambda t} = 1200 e^{-(0.0462 \times 30)} = 300.84\ \text{dis/ min}$$

$$\text{Volume of blood} = \frac{300.84}{0.5} = 601.68\ cm^3$$

Trial Questions

1. A radioactive element P is present as 1% of the atoms of a mono-atomic element Q which has an atomic mass of 80. Calculate the;

i Number of atoms of P in 20g of Q.

ii. Rate of decay of P after 25s. [Half-life = 10s]

[Ans: $1.505 \times 10^{19} atoms$, $1.844 \times 10^{19} s^{-1}$]

2. The activity of the sample of dead is 10 counts per minute while activity for a living plant is 19 counts per minute. If the half-life of carbon-14 is 5500 years, find the age of the wood sample. [Ans: 5094.08 years]
3. A radioactive source emits 2 x 10^5 alpha particles per second. The particle produces a saturation current of 1.1 x 10^{-8}A in an ionization chamber. If the energy required to produce an ion pair is 32eV, determine the energy in MeV of an alpha particle emitted by the source.

   [Ans: 11MeV]
4. A silver isotope ${}^{108}_{47}Ag$ has a half-life of 2.4 minutes. Initially, a sample contains 2 x 10^6 nuclei of ${}^{108}_{47}Ag$. Find the number of radioactive nuclei after 1.2 minutes.

   [Ans: $1.414 \times 10^6 nuclei$]
5. 1 kg of wood from shipwreck has an activity of 120 counts per second due to ^{14}C, whereas the same amount of wood had an activity of 200 counts per second. Find the age of the shipwreck. [Half-life ^{14}C = 5700 years] [Ans: $4.202 \times 10^3 years$]
6. A radioactive source contains 1.0μg of plutonium of mass number 239. If the source emits 2300 alpha particles per second, calculate the half-life of plutonium.

   [Ans: $7.589 \times 10^{11} s$]
7. The radioisotope ${}^{90}_{38}Sr$ decays by emission of beta particles. The half-life of the radioisotope is 28.8 years. Determine the activity of 1g of the isotope. [Ans: 1.609×10^{20} $per\ year$]
8. The activity of a sample of wood is 50 counts per hour while activity of a living plant is 100 counts per minute. If the half-life of carbon-14 is 5565 years. Find the age of the wood sample.

   [Ans: 38896.5 years]
9. Consider the nuclear reaction below

   ${}^{235}_{92}U + {}^{1}_{0}n \rightarrow {}^{x}_{56}Ba + {}^{89}_{y}Kr + 3{}^{1}_{0}n + Energy.$
   (i) Find the value of x and y.
   (ii) Calculate the energy released by one mole of ${}^{235}_{92}U$ in the above reaction.
   (iii) Explain why neutrons are preferred to charged particles for inducing nuclear reactions. [${}^{235}_{92}U = 235.117U$, ${}^{1}_{0}n = 1.0090U$, ${}^{x}_{56}Ba = 143.9577U$, ${}^{89}_{y}Kr = 88.926U, 1U = 931MeV$]

   [Ans: $= 144, y = 136$, $1.207 \times 10^{26} MeV$]

10. Calculate the binding energy per nucleon for ${}^{56}_{26}Fe$ nucleus.

    [${}^{1}_{0}n = 1.008665U, {}^{1}_{1}P = 1.007277U, {}_{-1}^{0}e = 5.4858 \times 10^{-4}U, {}^{56}_{26}Fe = 55.9349, 1U =$

$931 MeV$] [Ans: 8.787 MeV]

11. (a). A nucleus of $^{37}_{17}Cl$ emits an alpha particle followed by two beta particles. Show that the final nucleus is an isotope of chlorine.

(b). Why is radioactivity not affected by chemical combination or changes in physical environment?

12. A nucleus of Uranium disintegrates to thorium (Th) with emission of an alpha particle. Given that mass of $^{238}_{92}U$ = 238.12494u, mass of Th = 234.11650u, mass of alpha particle = 4.00387u,

i. Write a balanced equation for the reaction above.

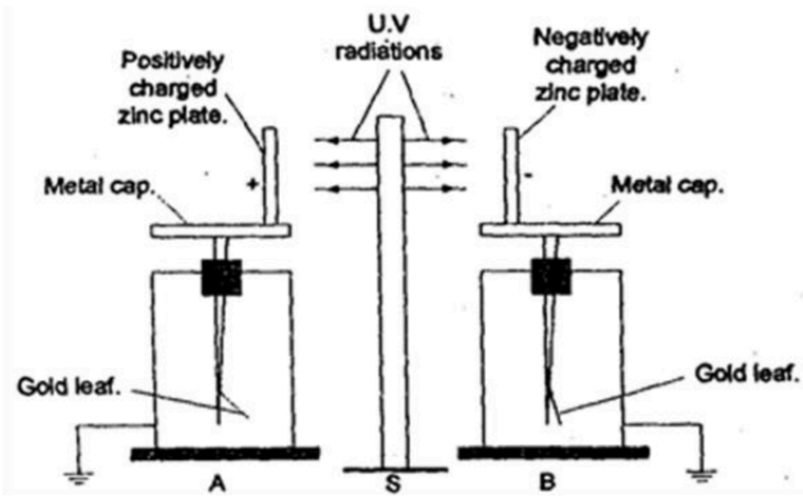
ii. Calculate the velocity of the alpha particle.

[Ans: $1.448 \times 10^{3} ms^{-1}$]

CHAPTER 21: PHOTO-ELECTRIC EFFECT

Photo-electric emission is the emission of electrons from the metal surface when electromagnetic radiations of a frequency greater than the threshold frequency of that metal, falls on it.

EXPERIMENT TO DEMONSTRATE PHOTOELECTRIC EFFECT



The set up is as shown above. Two clean zinc plates are charged, one positively and the other negatively. The positively charged plate is placed on an uncharged electroscope A while the negatively charged plate is placed on a placed on an uncharged electroscope B as shown in figure above. Both electroscopes are earthed and a source S is placed between them to illuminate the plates with ultra violet radiations.

Observation: It will be observed that the leaf of electroscope A remains divergent while that of electroscope B falls slowly.

Explanation: The leaf of electroscope A remains divergent because the electrons emitted by the plate on it, when u.v radiation falls on the plate, are attracted back by the positive charges of the plate. Thus there's no loss of charge in this case. On the other hand, the leaf of electroscope B falls slowly because when u.v radiation falls on plate placed on it, the emitted electrons will be repelled further by the negative charges of the plate. Thus the plate losses charge and the leaf falls slowly

Deduction: Photoelectric emission takes place on the negatively charged plate placed on electroscope B but doesn't take place on the positively charged plate placed on electroscope A.

LAWS OF PHOTOELECTRIC EMISSION

1st law: There's no time lag between irradiation and emission of photoelectrons.

2nd law: The number of electrons emitted per second (photo-current) is proportional to the intensity of incident radiation.

3rd law: There's a minimum frequency called threshold frequency below which no photoelectric emission occurs however much the intensity is.

4th law: The maximum kinetic energy of electrons is directly proportional to the frequency but independent of the intensity of radiation.

Explanations of the above laws of photoelectric emission.

Explanation of the 1st law: Each electron needs one photon for it to leave the metal surface. If the photon is less than the work function, it's rejected. But if the electron gains *a* photon of the right energy, it will immediately leave the surface.

Explanation of the 2nd law: When intensity increases, the number of photons reaching the surface per second increases which increases the number of electrons per second.

Explanation of the 3rd law: Intensity only increases the number of photons per second but doesn't increase their energy. Therefore, when the photons have a frequency less than the threshold frequency, they will have energy less than the work function of the metal surface and will be rejected. Hence no photoelectric emission takes place.

Explanation of the 4th law: When the frequency is increased, electrons will gain an energy (hf) which will exceed the work function (hf_0). The difference in these two energies is used as kinetic energy. Therefore, as frequency (f) increases, the difference $(hf - hf_0)$ increases hence increasing kinetic energy.

How wave theory fails to account for photoelectric effect

Instantaneous emission: From wave theory, radiation energy is uniformly spread over the whole wave front and thus an electron absorbs only a fraction of the total energy. This implies that there should be a time lag between irradiation and emission of photoelectrons which is not the case in photoelectric emission.

Existence of threshold frequency: Wave theory predicts continuous absorption of and accumulation of energy. Therefore, radiation of high enough intensity should be able to cause emission eve when the frequency is below the threshold frequency which is not the case in photoelectric emission.

Variation of kinetic energy: From wave theory, increasing intensity would mean more energy and hence greater value of maximum kinetic energy. However, maximum kinetic energy in photoelectric emission, kinetic energy depends on the frequency of incident radiation and not on intensity.

APPLICATIONS OF A PHOTOCELL

(a) As a photoelectric cell

i. Used to power solar watches,
ii. Used to power solar panels.
iii. Used to power calculators.

How a photoelectric cell is used

When the radiation falls on say a solar watch, electrons are emitted and these electrons cause a, current to flow through the watch and it begins to operate.

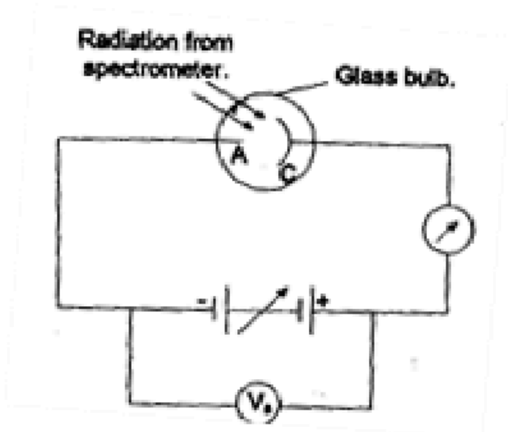
(b) As a photo emissive cell

i. Used to detect intruders.
ii. Used in switches to open doors,
iii. Can be connected to an automatic switch to switch on light.

How a photo emissive cell is used

For the case of detection of an intruder, the intruder intercepts infra-red beam falling on a photo cell hence cutting off current. The interruption therefore sets the alarm off.

EXPERIMENT TO DETERMINE PLANK'S CONSTANT OF PHOTOELECTRIC EFFECT

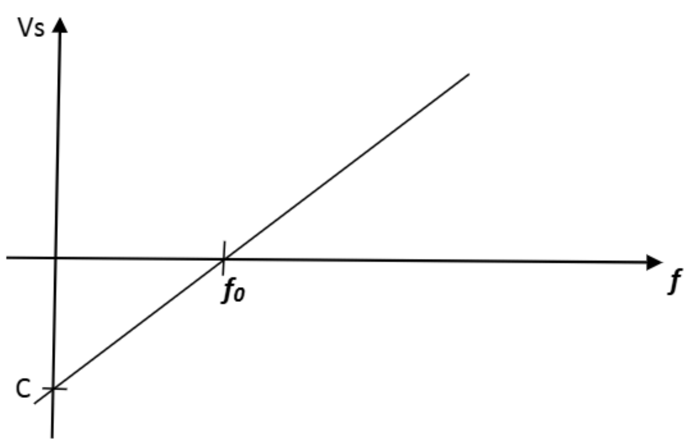


The apparatus is setup as shown above with a variable voltage source connected as shown above. Radiation of known frequency from a spectrometer is made incident on the cathode C of photocell.

The voltage across the cathode C and anode A is varied until the photo-current is zero. The stopping potential V_s is noted.

Measurement of V_s is repeated using other frequencies, f of radiation.

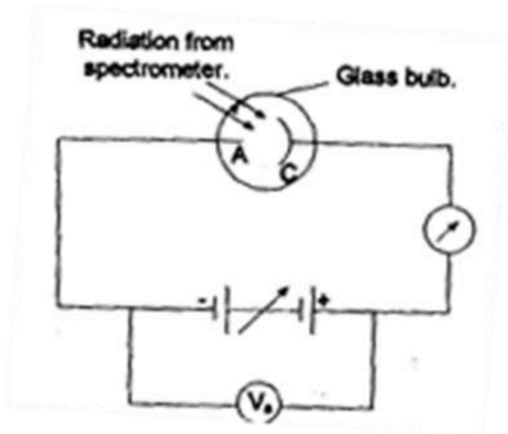
A graph of V_s against f is plotted as shown below. The slope, S of the graph is then obtained.



$$Slope, S = \frac{h}{e}$$

Therefore plank's constant, $h = Se$

EXPERIMENT TO VERIFY EINSTEIN'S EQUATION FOR THE PHOTOELECTRIC EFFECT



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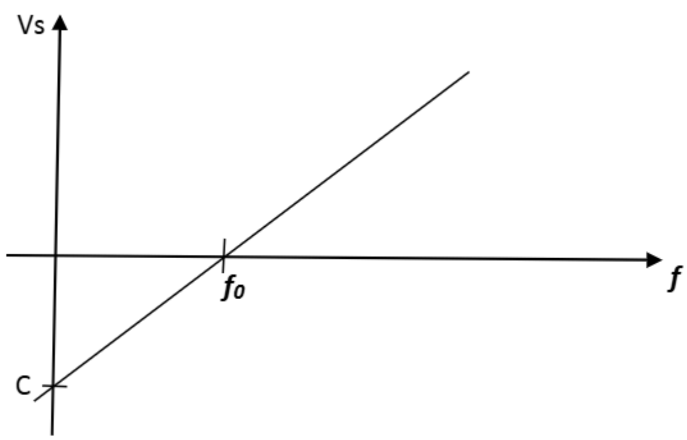


From the general equation of a straight line, $y = mx + c$ where c is the intercept on the V_s axis

$$m = slope = \frac{h}{e}\ , y = V_s,\ \ and\ x = f$$

$$y = mx + c$$

$$V_s = \left(\frac{h}{e}\right) f + c$$

When $V_s = 0, f = f_0$

$$0 = \left(\frac{h}{e}\right) f_0 + c$$

$$c = -\left(\frac{h}{e}\right) f_0$$

Therefore,

$$V_s = \left(\frac{h}{e}\right) f - \left(\frac{h}{e}\right) f_0$$

$$eV_s = h(f - f_0)$$

This is Einstein's equation and it's equal to the maximum kinetic energy of a photoelectron.

QUANTUM THEORY

It states that light is emitted or absorbed in discrete packets of energy called photons. According to quantum theory, when light is incident on a metal surface, each photon interacts with only one electron on the surface of the metal giving it all its energy. A photon is absorbed if its energy is greater than the work function of the metal and rejected if its energy is less than the work function of the metal. Part of the energy (work function) is used to overcome the nuclear attraction of the electron and the rest becomes kinetic energy of the photoelectrons. Therefore, the electron can escape from the metal surface.

$$hf = \omega_0 + \frac{1}{2} m_e {v_{max}}^2$$

where (hf) is the energy of the incident radiation

NOTE: for an electromagnetic radiation, $c = f\lambda$ Using this in the above equation gives;

$$\frac{hc}{\lambda} = \omega_0 + \frac{1}{2} m_e {v_{max}}^2$$

Experimental evidence of quantum theory of matter

(i) Optical spectra: An excited electron in the excited state jumps directly from higher to lower energy levels and in the process emits energy of a certain wavelength which forms a line. This line in the ' optical emission spectrum indicates presence of a frequency of light; hence a photon.

(ii) **X-ray line spectra:** Transition of electrons from one shell to another leads to liberation of energy in discrete packets characteristic of the metal target. These packets are the photons.

(iii) **Photoelectric emission:** To liberate an electron from the surface of a metal, a quantity of energy called work function has to be supplied. This is a photon.

How photoelectric emission provides evidence for quantum theory of light

(1) Increasing intensity of light only increases the number of photons per second but the energy of each photon remains the same. So maximum kinetic energy is independent of the intensity which is true according to experimental observations.

(2). Increasing intensity of light increases the number of photons striking the surface per second which increases the rate of emission of electrons from the surface. This implies that photon-current increases with increase in intensity which is true according to experimental observations.

(3). Increasing frequency increases photon-energy (energy of each photon). Therefore, maximum kinetic energy increases with increase in frequency which is true according to experimental observations.

TERMS AND DEFINITIONS

(a) Work function: this is the minimum amount of energy which must be supplied to a metal surface for it to discharge the outermost electron. It can also be defined as the minimum amount of energy required for an electron to overcome the forces of attraction of the nuclei on the surface of the metal.

(b) Threshold/cut-off frequency: this is the minimum frequency required to have electrons emitted from the surfaces of the metals.

(c) Stopping potential: this is the least negative potential that can stop all the electrons emitted at the cathode by the photocell.

(d).Plank's theory

It states that the energy of a photon is directly proportional to its frequency.

$$E \propto f$$

$$E = hf$$

Where h is plank's constant.

WORKED EXAMPLES

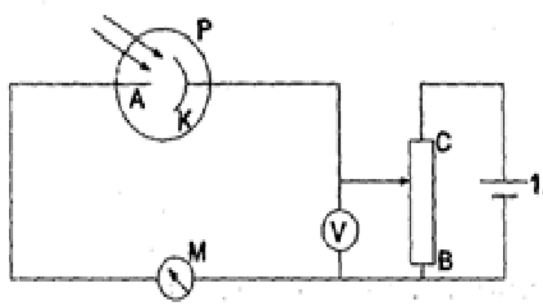
Where applicable, use the following constants

Charge of an electron, $e = 1.6 \times 10^{-19}$

Plank's constant, $h = 6.63 \times 10^{-34} Js$

Mass of an electron, $m_e = 9.11 \times 10^{-31} kg$

1.



P is a vacuum photocell with anode, A and cathode, K made from the same metal with work function 2eV. The cathode is illuminated by monochromatic light of constant intensity and of wavelength 4.4×10^{-7} m.

a (i) Describe and explain how the current shown by the micro-ammeter, M will vary as the slider of the potential divider is moved from B to C.

(ii) What will be the reading of the high resistance voltmeter, V when the photo-electric emission just ceases?

(b) With the slider half way between B and C, describe and explain how the reading of the ammeter, M would change if;

(i) the intensity of the light was increased,

(ii) the wavelength of the light was changed to 5.5×10^{-7} m.

Solution

(a) (i) The cathode, K has a positive potential relative to the anode, A. As the slider moves from B to C, the cathode becomes more positive with respect to the anode.

A retarding force is therefore experienced by the electrons as they move to the anode until a point is reached when no electrons reach the anode. At this point, no current flows in the photocell. This will make the reading of the micro-ammeter to decrease to zero.

(ii) $\omega_0 = 2eV = 2 \times 1.6 \times 10^{-19} = 3.2 \times 10^{-19} J, \lambda = 4.4 \times 10^{-7} m$

$$\frac{hc}{\lambda} - \omega_0 = eV_s$$

$$\frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{4.4 \times 10^{-7}} - 3.2 \times 10^{-19} = (1.6 \times 10^{-19}) V_s$$

$(1.6 \times 10^{-19})V_s = 1.32 \times 10^{-19}$

$V_s = 0.8253\ V$

(b) (ii) When the slider is midway between B and C, the p.d in the photocell is $\left(\frac{1}{2} \times 1.5\right)$= 7.5V.Since this p.d is less than the stopping potential, there will be photoelectrons reaching the anode.

Increasing the intensity increases the number of photons reaching the cathode per second which in turn increases the number of electrons emitted at the cathode per second. This will increase the reading of M.

(ii) $\omega_0 = \frac{hc}{\lambda_0}$

$$\lambda_0 = \frac{6.63\times10^{-34}\times3\times10^{8}}{3.2\times10^{-19}} = 6.216 \times 10^{-7}m$$

Therefore

$$\lambda_1 = 4.4 \times 10^{-7}m, \lambda_0 = 6.216 \times 10^{-7}, \lambda_2 = 5.5 \times 10^{-7}m$$

$$\lambda_1 < \lambda_2 < \lambda_0$$

The new wavelength (λ_2) is longer than the previous one (λ_1) but less than the threshold value (λ_0). Electrons will therefore be emitted but with a lower kinetic energy since $\lambda_2 > \lambda_1$. The number of electrons emitted per second also remains the same since the intensity is kept constant. The reading of M will therefore remain constant.

2. When light of wavelength 5.9 x 10^{-7} m is incident on sodium metal, electrons of maximum kinetic energy 1.71 x 10^{-20}J are emitted. Calculate the maximum kinetic energy of the electrons that will be emitted by sodium metal illuminated by light of wavelength 4.5 x 10^{-7} m.

Solution

$$\lambda_1 = 5.9 \times 10^{-7}m, K.E_1 = 1.71 \times 10^{-20}J,\ \lambda_2 = 4.5 \times 10^{-7}m, K.E_2 = ?$$

$$K.E_1 = \frac{hc}{\lambda_1} - \omega_0$$

$$1.71 \times 10^{-20} = \frac{6.63\times10^{-34}\times3\times10^{8}}{5.9\times10^{-7}} - \omega_0$$

$$\omega_0 = 3.371 \times 10^{-19} - 1.71 \times 10^{-20} = 3.2 \times 10^{-19}J$$

$$K.E_2 = \frac{hc}{\lambda_2} - \omega_0$$

$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{4.5 \times 10^{-7}} - 3.2 \times 10^{-19} = 1.22 \times 10^{-19} J$$

3. The work function of tungsten is 4.49eV. Ultraviolet radiation of wavelength 250nm falls on the surface. Calculate:
   i. The cut off wave length of photo emission.
   ii. The stopping potential

Solution

$$\omega_0 = 4.49 \times 1.6 \times 10^{-19} = 7.184 \times 10^{-19} J, \lambda = 250 \times 10^{-9} m$$

$$\lambda_0 = \frac{hc}{\omega_0}$$

$$\lambda_0 = \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{250 \times 10^{-9}} = 2.679 \times 10^{-7} J$$

(ii) $$\frac{hc}{\lambda} - \omega_0 = eV_s$$

$$\frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{250 \times 10^{-9}} - 7.184 \times 10^{-19} = (1.6 \times 10^{-19}) V_s$$

$$(1.6 \times 10^{-19}) V_s = 7.72 \times 10^{-20}$$

$$V_s = 0.4825\ V$$

4: Violet light of length 0.4μm is incident on a metal surface of threshold wavelength 0.65μm. Find the maximum speed of the emitted electrons.

Solution

$$\lambda_0 = 0.65 \times 10^{-6} m, \lambda = 0.4 \times 10^{-6} m, K.E_2 = ?$$

$$\frac{1}{2} m_e {v_{max}}^2 = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$${v_{max}}^2 = \frac{2hc}{m_e}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$${v_{max}}^2 = \frac{2 \times 6.63 \times 10^{-34} \times 3 \times 10^{8}}{9.11 \times 10^{-31}}\left[\frac{1}{0.4 \times 10^{-6}} - \frac{1}{0.65 \times 10^{-6}}\right]$$

$${v_{max}}^2 = 4.199 \times 10^{-11}$$

$$v_{max} = \sqrt{4.199 \times 10^{-11}} = 6.48 \times 10^{-5} ms^{-1}$$

Trial questions

1. Calculate the maximum speed of the photo electrons emitted by a caesium surface irradiated with light of wavelength 484nm if the work function of caesium is 3 x 10^{-19}J
   [Ans: $4.935 \times 10^{5} ms^{-1}$]
2. A 100mW beam of light of wavelength 4 x 10^{-7}m falls on a caesium surface of a photocell.

i. How many electrons strike the caesium surface per second?

ii. If 80% of the photons emit photo electrons, find the resulting photo current.

iii. Calculate the kinetic energy of each photo electron, if the work function of caesium is 2.15eV [Ans: $2.011 \times 10^{17} s^{-1}, 2.574 \times 10^{-12} A, 1.533 \times 10^{-19} J$

3. A source emits monochromatic light of frequency at a rate of 0.1W. Of the photons given out, 0.15% falls on the cathode of the photocell which gives the current of 6μA in an external surface. Assuming that this current consists of all photo electrons emitted, calculate;

i. The energy of the photon.

ii. The number of photons leaving the source per second.

iii. The percentage of the photons falling on the cathode which produce photo electrons.

[Ans: $3.647 \times 10^{-19} J$,$2.742 \times 10^{17}\ s^{-1}$, 9.115%]

4. When a certain metallic monochromatic radiation of length *X* the maximum kinetic energy of photo electrons released from the surface is 30eV. When the same surface is illuminated with radiation of wavelength 2λ, the maximum kinetic energy of the photo electrons is 10eV. Show that the maximum wavelength of incident radiation on this metallic surface that can cause the release of the electrons is given by $\lambda_{max} = 4\lambda$

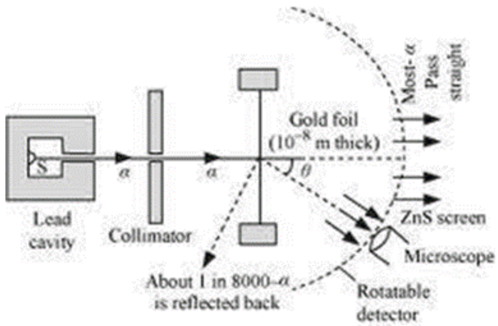
5. The work function of a metal is 2eV. Calculate the stopping potential when illuminated by a light of frequency 6 x 10^{14} Hz. [Ans: 0.486 V]

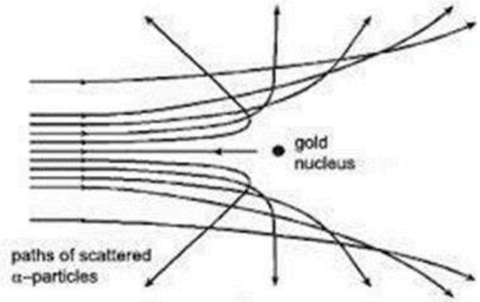
CHAPTER 22: THE ATOM

RUTHERFOLD'S GOLD FOIL EXPERIMEMT

This was an experiment carried out by Ruther fold to investigate the scattering of alpha particles by thin films of gold.

Scattering of alpha particles





A thin piece of gold was placed perpendicular to the source of alpha particles. A narrow spot beam of alpha particles from the source inside a metal block was incident on a thin gold foil. Whenever a particle hit the screen, it produced a faint flash of light called scintillation.

The experiment was carried out in a dark room and the scintillations were observed through the microscope. The screen could be rotated about the metal foil and by counting the number of scintillations in various positions in equal intervals of time, the angular dependence of the scattering was determined.

Since the range of alpha particles in air is limited to about 5cm, the apparatus was evacuated so that particles would not be prevented from reaching the screen.

Observations

it was observed that;

(i) Most of the alpha particles passed through the foil undeflected.

(ii) A few particles were scattered through small angles.

(iii) Very few particles were deflected through angles greater than 90°.

Conclusions

He concluded that;

(i) It the gold foil atoms were completely solid, no alpha particle would penetrate through to the other side of the gold foil,

(ii) Since the majority passed through the gold foil undeflected, then most of the space inside the atom is empty,

(iii) The scattering of the alpha particles was due to head on collisions of the alpha particles (positively charged) with the atoms of gold (negatively charged)

(iv) The small deflection was due to repulsion by the positive charge.

Rutherford's model of an atom

(i) An atom consists of a positive charge confined to the centre where most of the mass is concentrated,

(ii) Electrons orbit round the nucleus like planets do round the sun.

(iii) It's the electron cloud that accounts for the volume of the atom.

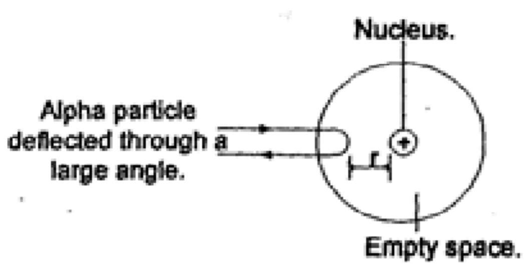
Rutherford's failures

(i) Much as electrons revolve round the nucleus, the do so only in certain allowed orbits and when they are in these orbits, they don't emit radiations. Rutherford failed to explain this.

(ii) Electrons can also jump from one orbit of energy say E_2 to another of lower energy E_1 and the difference is emitted as one quantum of frequency *f* given by plank's equation of frequency $E_2 - E_1 = hf$. Rutherford failed to explain this.

Closest distance of approach of an alpha particle to the nucleus

An alpha particle directed onto the nucleus is deflected through a large angle and the smallest distance alpha particles can reach the nucleus is referred to as the closest distance of approach of the alpha particles to the nucleus.



THE BOHR ATOM

A Bohr atom one whose centre (the nucleus) is surrounded by electrons moving in definite circular or elliptical orbits and when in these stable orbits, the atom doesn't radiate.

Bohr's postulate

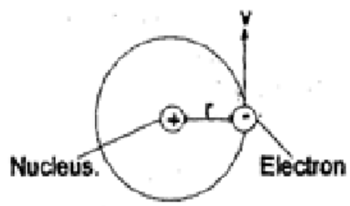
(i) Electrons can revolve round the nucleus only in certain allowed orbits and when in these orbits, they don't emit any radiations.

(ii) An electron can jump from an orbit of higher energy E_2 to that of a lower energy E_1 and a radiation is emitted of frequency, f given by $\Delta E = hf, where\ \Delta E = E_2 - E_1$.

(iii) The angular momentum of the electrons are whole number multiples of $\frac{h}{2\pi}$ where h is a constant known as plank's constant.

Bohr's postulate applied to a hydrogen atom

Consider an electron of mass m and charge e revolving round the nucleus of an atom with a velocity v and in an orbit of radius r as shown below.



Expression for the kinetic energy of the electron

The nucleus of the atom has a charge equal in magnitude to that of an electron. Therefore, by coulomb's law, electrostatic force exerted on an electron is given by;

$$F_1 = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

Centripetal force on the electron directed towards the nucleus is also given by;

$$F_2 = \frac{mv^2}{r}$$

At equilibrium, $F_2 = F_1$. Therefore,

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

$$mv^2 = \frac{e^2}{4\pi\ \varepsilon_0 r}$$

Thus, kinetic energy of the electron is given b

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}\left[\frac{e^2}{4\pi\ \varepsilon_0 r}\right]$$

$$K.E = \frac{e^2}{8\pi\varepsilon_0 r} \quad \ldots\ldots\ldots..(i)$$

Expression for the potential energy of the electron

Potential energy is equivalent to the work done in moving an electron from infinity to an orbit of radius r with the nucleus as its centre.

The small work done in moving an electron a small distance δr in the electron cloud is given by;

$$\delta W = F\delta r = \left(\frac{e^2}{4\pi\varepsilon_0 r^2}\right)\delta r$$

Therefore, potential energy of the electron is given by;

$$\text{P.E} = \int_{\infty}^{r}\left(\frac{e^2}{4\pi\varepsilon_0 r^2}\right)dr = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\left[\frac{-1}{r}\right]_{\infty}^{r} = \frac{-e^2}{4\pi\varepsilon_0 r}$$

Thus, potential energy of the electron is given by;

$$\text{P.E} = \frac{-e^2}{4\pi\varepsilon_0 r} \quad \ldots\ldots\ldots\ldots(ii)$$

Expression of the total energy of the electron

$$\begin{pmatrix} total \\ energy \end{pmatrix} = \begin{pmatrix} kinetic \\ energy \end{pmatrix} + \begin{pmatrix} potential \\ energy \end{pmatrix}$$

$$E_n = \left[\frac{e^2}{8\pi\varepsilon_0 r} + \frac{-e^2}{4\pi\varepsilon_0 r}\right] = \frac{-e^2}{8\pi\varepsilon_0 r}$$

Thus, total energy of the electron is given by;

$$E_n = \frac{-e^2}{8\pi\varepsilon_0 r} \quad \ldots\ldots\ldots.(iii)$$

Expression of total energy using Bohr's postulates

If *h* is a constant the plank's constant, and *n* is an integer called the principal quantum number.

Then according to Bohr's postulates,

$$\begin{pmatrix} angular \\ momentum \end{pmatrix} \propto \left(\frac{h}{2\pi}\right)$$

$$mvr \propto \frac{h}{2\pi}$$

$$mvr = \frac{nh}{2\pi}$$

$$v = \frac{nh}{2\pi mr}$$

Thus, kinetic energy of the electron is given by;

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{nh}{2\pi mr}\right)^2 = \frac{1}{8m}\left(\frac{nh}{\pi r}\right)^2$$

$$K.E = \frac{1}{8m}\left(\frac{nh}{\pi r}\right)^2 \quad \ldots\ldots\ldots(iv)$$

Equating equations (i) and (iv) gives

$$\frac{e^2}{8\pi\varepsilon_0 r} = \frac{1}{8m}\left(\frac{nh}{\pi r}\right)^2$$

$$\frac{e^2}{\varepsilon_0} = \frac{n^2h^2}{\pi mr}$$

$$r = \frac{n^2h^2\varepsilon_0}{\pi me^2} \dots\dots\dots (v)$$

Substituting for r in equation (iii) gives;

$$E_n = \frac{-e^2}{8\pi\varepsilon_0 r} = \frac{-e^2}{8\pi\varepsilon_0}\left[\frac{\pi me^2}{n^2h^2\varepsilon_0}\right] = \frac{1}{n^2}\left(\frac{-me^4}{8h^2{\varepsilon_0}^2}\right)$$

$$E_n = \frac{1}{n^2}\left(\frac{-me^4}{8h^2{\varepsilon_0}^2}\right)$$

Thus, $E_n \propto \frac{1}{n^2}$

NOTE: E_n is negative because electrons are bound to the nucleus of the atom, so work must be done to remove the electron from the atom to infinity where the energy is considered to be zero and this work is done against the attraction binding the electrons in the atom.

Frequency of radiation during electron transition

From Bohr's postulates, an electron can jump from an orbit of higher energy E_2 to that of a lower energy E_1 and a radiation is emitted of frequency f given by $\Delta E = E_2 - E_1$

Since $E_n \propto \frac{1}{n^2}$, then

$$\Delta E = E_2 - E_1 = \left(\frac{-me^4}{8h^2{\varepsilon_0}^2}\right)\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$$

$$hf = \left(\frac{-me^4}{8h^2{\varepsilon_0}^2}\right)\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$$

$$f = \left(\frac{-me^4}{8h^3{\varepsilon_0}^2}\right)\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$$

$f = a\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$ where $a = \left(\frac{-me^4}{8h^3{\varepsilon_0}^2}\right)$ and a is a constant

Also $f = \frac{c}{\lambda}$

Therefore, $\frac{c}{\lambda} = a\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$

$$\frac{1}{\lambda} = \frac{a}{c}\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$$

$$\frac{1}{\lambda} = R_H\left[\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right]$$

Where $R_H = \frac{a}{c}$ $, and\ c\ is\ the\ velocity\ of\ light\ in\ vacuum$

Bohr's assumptions

i. Each electron moves in a circular orbit which is centered on the nucleus,

ii. The necessary centripetal force is provided by the centripetal force of attraction between the positively charged nucleus and the negatively charged electron.

Failures of Bohr's postulate

i. It can only explain spectra for simpler atoms with few electrons such as hydrogen.

ii. It also can't explain the fine structure of spectral lines of hydrogen

iii. It says electron orbits are circular yet they are elliptical.

ENERGY LEVELS OF A HYDROGEN ATOM

According to Bohr's model of an atom, electrons are arranged in permitted orbits of defined amount of energy. These orbits are known as energy levels of the atom. The levels can be represented by horizontal lines arranged one above the other to form an energy level with each line indicating a particular energy value.

An electron can jump from one energy level to another by gaining or losing energy. All levels have negative values because the energy of an electron at rest outside an atom is taken to be zero and when the electron falls into the atom, energy is lost as electromagnetic radiation.

The hydrogen spectrum is obtained by using different numbers of n when calculating the energy levels. The lowest energy level is obtained when n=l. The u.v series are obtained when the energy level falls to the lowest energy level corresponding to n=l. The visible spectrum is obtained for energy level corresponding to n=2.The infrared spectrum is obtained when n=3.

The energy change E for the u.v series is greater than for the visible spectrum. Since E= *hf*, the frequency of the u.v radiation is greater than that of the visible radiation. Since $c = f\lambda$, the u.v wavelengths are shorter than those in the-visible spectrum. The lowest energy level E_1 is called the ground state.

By definition, the energy required to just remove the electron from the ground state to infinity (E_∞) is called the **ionization energy** (E_{ion})

$$E_{ion} = 0 - E_1$$

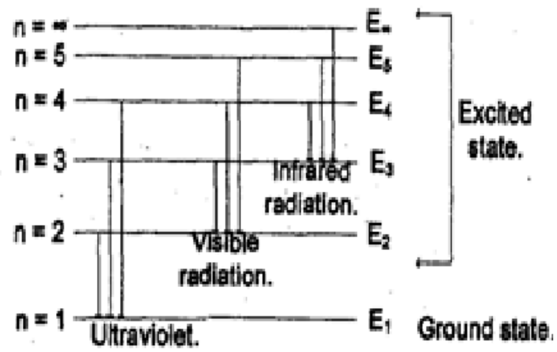
But $E_1 = -13eV \quad thus, \;\; E_{ion} = 13eV$

WORKED EXAMPLES

Where applicable, use the following constants

1. Draw an energy level diagram for hydrogen to indicate emission of ultra-violet, visible and infra-red spectral lines.

Solution



2. An alpha particle with kinetic energy 5MeV makes head on collision with an atom of gold in the gold foil. It's deflected through 180°. If the atomic number of gold is 79, calculate the distance of nearest approach of the alpha particle to the nuclear centre of gold and what is the significance of the result? $[\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 F^{-1}m]$

Solution

$$K.E = 5MeV = 5 \times 10^6 \times 1.6 \times 10^{-9} = 8 \times 10^{-13}J \ , \ Z = 79 \ , \ m = 4$$

$$r = \frac{4ke^2Z}{mu^2}$$

$$mu^2 = \frac{4ke^2Z}{r} \quad \ldots\ldots\ldots\ldots (i)$$

$$K.E = \frac{1}{2}mu^2$$

$$8 \times 10^{-13} = \frac{1}{2}\left(\frac{4ke^2Z}{r}\right)$$

$$r = \frac{1}{2}\left(\frac{4ke^2Z}{8\times10^{-13}}\right)$$

$$r = \frac{1}{2}\left(\frac{4\times9\times(1.6\times10^{-9})^2\times79}{8\times10^{-13}}\right) = 4.55 \times 10^{-14}m$$

Significance: The result is the estimation of the radius of the nucleus, i.e. the radius of the nucleus is just less than 4.55 x 10^{-14}m.

3. The energy levels of a hydrogen atom are given by $E_n = \frac{-21.7\times10^{-9}}{n^2}$ joules, where n takes on the values 1, 2, 3, …. Find the shortest wave length of radiation which can be emitted by the

hydrogen atom.

Solution

Minimum wavelength is the transition from E_∞ to E_1

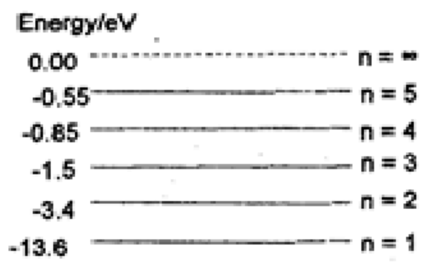
$$E_\infty - E_1 = \frac{hc}{\lambda}$$

$$\frac{-21.7\times10^{-9}}{\infty^2} - \frac{-21.7\times10^{-9}}{1^2} = \frac{6.63\times10^{-34}\times3\times10^{8}}{\lambda}$$

$$21.7\times10^{-9} = \frac{1.989\times10^{-25}}{\lambda}$$

$$\lambda = \frac{1.989\times10^{-25}}{21.7\times10^{-19}} = 9.166\times10^{-8}m$$

4. The diagram below shows some energy levels of the hydrogen atom



(i) Use the diagram to explain the emission spectrum of hydrogen.

(ii) Calculate the speed of an electron which could just ionize the hydrogen atom.

(iii) Calculate the minimum wavelength of the hydrogen spectrum and state the region of the electromagnetic spectrum in which it lies.

Solution

(i) When atoms of hydrogen are excited by say heat, the electrons make transitions to higher energy levels. The atoms become unstable since energy has increased. Electron transition occurs to a vacancy left in the lower energy levels and a radiation of definite wavelength or frequency is emitted. A series of well-defined separated bright lines of definite wave length or frequency are formed against a background and this is an emission line spectrum.

(ii) The energy that could just ionize the hydrogen atom is the first ionization energy.

$$\frac{1}{2}m_e u^2 = -E_1$$

$$\frac{1}{2}\times 9.11\times10^{-31}\times u^2 = -(-13.6\times1.6\times10^{-19})$$

$$u^2 = \sqrt{\left(\frac{13.6\times1.6\times10^{-19}}{0.5\times9.11\times10^{-31}}\right)} = 2.186\times10^{6}ms^{-1}$$

(iii) Minimum wavelength is the transition from E_∞ to E_1

$$E_\infty - E_1 = \frac{hc}{\lambda}$$

$$[0-(-13.6)] \times 1.6 \times 10^{-19} = \frac{6.63\times10^{-34}\times3\times10^{8}}{\lambda}$$

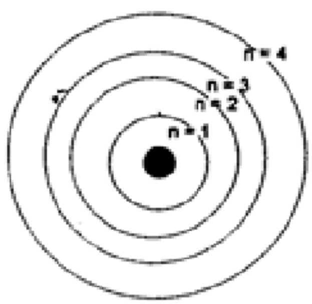
$$2.176 \times 10^{-18} = \frac{1.989 \times 10^{-25}}{\lambda}$$

$$\lambda = \frac{1.989\times10^{-25}}{2.176\times10^{-18}} = 9.141 \times 10^{-8}m$$

It lies in the ultra violet region of the electromagnetic spectrum

Trial questions

1. (a) The diagram below depicts possible electron orbits in the Bohr model of a hydrogen atom. Assume the orbits are circular



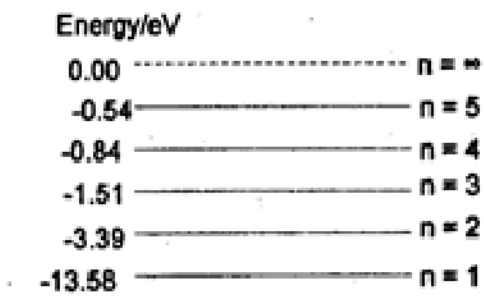
(i) Show that the total energy in an orbit of radius, r is given by $E = \frac{-e^2}{8\pi\varepsilon_0 r}$ where e is the electron charge.

(ii) If only the orbits allowed for the stable electron orbit are those where by $r = \frac{nh}{2\pi}$, where m is the mass of electron, v is the electron speed, n is an integer and h is plank's constant; show that the total energy in a(i) above can be expressed as $E_n = \frac{-me^4}{8h^2\varepsilon_0{}^2n^2}$

(b) (i) What is the significance of the negative sign in the expression of E in a(i) above?

(ii) Calculate the wavelength of the radiation that will be emitted when the electron makes a transition from n = 4 to n = 3. [Ans: $1.086 \times 10^{-6}m$]

2. The diagram below shows some energy levels of the hydrogen atom



(i) Why are the energy levels labeled with negative energies?

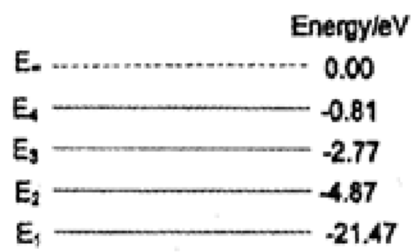
(ii) State which transmission will result in the emission of radiation of wavelength 487nm. Justify your answer by a suitable calculation. In which part of the electromagnetic spectrum does it lie?

(iii) Calculate the ionization potential.

(iv) Calculate the highest frequency in the Lyman series of the spectral lines.

[Ans: From n =4 to n=2 , 13.58V, $3.277 \times 10^{15} Hz$]

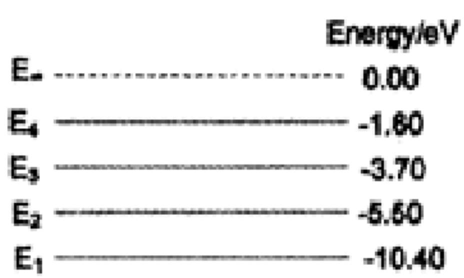
3. The diagram below shows some energy levels of neon



Determine the wavelength of radiation emitted in the electron transition from E_4 to E_3. In what region of the electromagnetic spectrum does the radiation line lie?

[Ans: $6.342 \times 10^{-7} m$]

4. The diagram below shows some of the energy levels of mercury atom.



Find the ionization energy of a mercury atom in joules. Find the wavelength of the radiation emitted when an electron moves from level 4 to level 2 and state the part of the electromagnetic spectrum where the radiation lies. [Ans: $1.664 \times 10^{-18} J,\ 3.188 \times 10^{-7} m$]

5.

The figure above shows a simplified energy level diagram for atomic hydrogen. A free electron with kinetic energy of 12eV collides with an atom of hydrogen and causes it to be raised to its first excited state. Calculate;

(i) The loss in kinetic energy of the free electron in eV after the collision

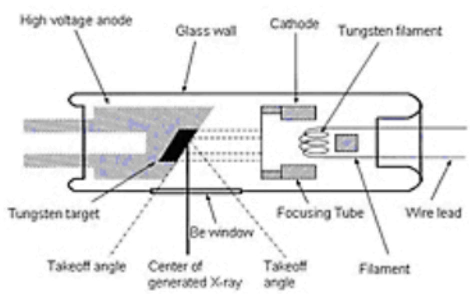
(ii) The wavelength of the photon emitted when the atom returns to the ground state.

[Ans: 1.8eV , $1.219 \times 10^{-7} m$]

CHAPTER 23: X-RAYS

X-rays are electromagnetic radiations/waves of very short wavelength produced when fast moving electrons are stopped by a hard metal target.

PRODUCTION OF X-RAYS



X RAY TUBE

The filament is heated by a low voltage and electrons are emitted from it by thermionic emission. The ejected electrons are focused on the target by the focusing cap and then accelerated by the extra high tension (E.H.T) connected across the terminals of the cathode and anode.

On striking the target, most of the energy (about 99%) of the electrons is converted to heat energy but a small percentage (1%) is converted into x-rays. The heat energy at the target is removed by the cooling fins. Little or no energy is lost by the electrons on their way to the target because of the vacuum.

Energy changes that take place during x-ray production

[Electrical energy from low a.c voltage]

↓

[[Heat energy from the filament]

↓

[Kinetic energy of moving electrons]

↓

[Heat energy in the target]

Features of an x-ray tube which makes it suitable for continuous x-ray production

- Low a.c voltage for heating,
- Source of electrons (heating filament).
- Accelerating p.d (E.H.T between cathode and anode).
- Target of high melting point (tungsten),
- Cooling system (cooling fins),
- A good conductor for anode (copper),
- Vacuum to ensure that little or no energy is lost by electrons on their way to the target.

Properties of x-rays

- They affect photographic plates,
- They travel in a straight line,
- They cause ionization of gases,
- They carry no charge,
- They can cause fluorescence.
- They can cause photoelectric emission,
- They can travel at a speed of light,
- They have a wave nature.
- They readily penetrate matter and are absorbed by very dense elements like lead.

PENETRATING POWER OF X-RAYS

This is the extent to which the ejected electrons penetrate matter (target). It's controlled by the p.d between the terminals of the cathode and the anode which determines the amount of kinetic energy with which the electrons strike the anode. The kinetic energy is proportional to this accelerating p.d. intensity remain constant.

That is; $K.\ E \propto eV,$ implying that $K.\ E = eV,$ where $K.\ E$ is kinetic energy is the accelerating p.d, and e is the electronic mass.

NOTE: If the accelerating voltage increases, kinetic energy of the electrons reaching the target increases but the number of electrons doesn't change. Heat dissipated in the target also increases.

HARD AND SOFT X-RAYS

X-rays with high penetrating power are known as hard x-rays. They have a short wavelength and are used to destroy cancer cells. On the other hand, x-rays with long wavelength and less penetrating power are known as soft x-rays. They are used in x-rays photography for human body.

INTENSITY OF X-RAYS

This is the power transmitted per unit area. It's controlled by the filament current which determines the number of electrons striking the metal target per second.

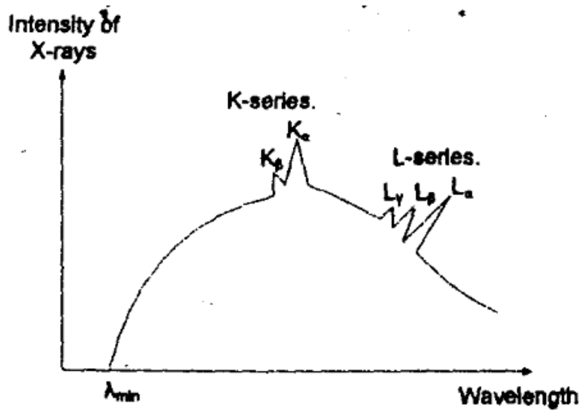
The greater the filament current, the greater the number of electrons striking the target per second, and hence the greater is the intensity. Penetrating power remains constant.

THE MAJOR DIFFERENCE BETWEEN X-RAY PRODUCTION AND PHOTOELECTRIC EMISSION

In photoelectric emission, electromagnetic radiation is incident on the metal surface which releases electrons and little energy is produced. However, in x-ray production, fast moving electrons strike the metal target producing x-rays (electromagnetic radiations) and a lot of heat is generated in the target. Therefore, photoelectric emission is the reverse of x-ray production.

ORIGIN OF CHARACTERISTIC X-RAY LINES FROM ATOMIC THEORY

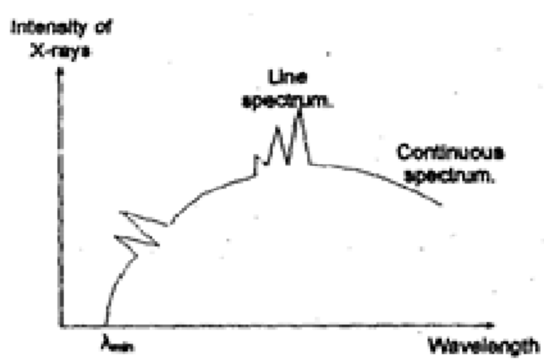
At very high voltages, it's possible for the bombarding electrons to penetrate deep into the atom and knock out an electron from the inner filled shell (k-shell). The knocked out electron can either be ejected completely out of the atom, or can occupy any of the higher unfilled shells. This puts the atom in an excited state and therefore unstable. As a result, there's electron transition from higher shells to the vacancy in the k-shell. An x-ray photon is then emitted whose energy is equal to the difference between the two energy levels and this results in k-series or lines. The transition that results into an electron ending in the L-shell results in the L-series. These are called characteristic x-ray lines.



GENERATION OF CONTINUOUS LINE SPECTRA OF X-RAYS IN AN X-RAYTUBE

Continuous spectrum is produced when electrons make collisions with the target atoms in which case they are decelerated. At each deceleration, x-rays of differing wavelengths are produced. These x-rays overlap each to form a continuous spectrum. The shortest wavelength x-rays are produced when electrons lose all their energy as x-ray photons in a single encounter with target atoms. The wavelength of the x-rays at this point is called cut-off wavelength. The x-rays with longer wavelength are as a result of electrons losing less than their total energy.

A GRAPH OF INTENSITY AGAINST WAVELENGTH OF X-RAYS

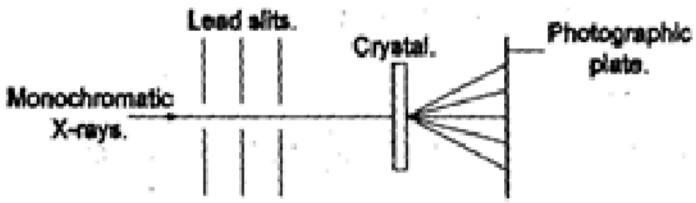


Minimum wavelength (λ_{min}): This is as a result of a bombarding electron having a head on collision with one target atom in a single encounter. The electron loses all its kinetic energy in form of an x-ray photon. This x-ray formed is the most energetic and has a maximum frequency hence minimum wavelength.

Continuous spectrum: This is due to the fact that a single electron undergoes a number of collisions before losing all its kinetic energy. At each collision, x-ray photons of smaller energies than maximum energy are given out. They overlap and therefore produce a continuous spectrum.

Line spectrum: some bombarding electrons can penetrate deep into the atom and displace electrons from their energy levels. The atoms become excited and unstable. Stability is restored by the electron falling from higher energy levels to lower vacant energy levels and x-ray photons are given out. These have definite frequencies thus forming x-ray spectrum with prominent lines. **NOTE:** The wave lengths of the prominent lines in a line spectrum are changed by altering the metal target since these lines are characteristic of the metal.

EXPERIMENT TO SHOW THE WAVE NATURE OF X-RAYS



A narrow beam of monochromatic x-rays is made incident on a crystal of dimensions comparable to the wavelength of x-rays behind, behind which is placed a photographic plate. Analysis shows that a central dark spot surrounded by a pattern of other smaller dark spots is formed. This shows that the x-rays are diffracted confirming their wave nature.

Reasons why x-rays are electromagnetic waves

- They give a line spectra.
- They eject electrons from matter by photoelectric emission and other methods. That is, they ionize gases.
- Their method of production involve accelerated charged particles.

INDUSTRIAL USES OF X-RAYS

- In art while analyzing the painting to find out whether the paintings are genuine or imitated,
- To detect cracks which are invisible to the eye in metal castings and welded joints.
- To detect defects in motor tyres,
- To study the structure of crystals.

BIOLOGICAL USES OF X-RAYS

- In medical examination of the human body to detect the complicated organs,
- To treat cancerous diseases and other malignant growth in the human body.
- To investigate the lungs in order to detect tuberculosis,
- To investigate the broken bones in x-ray photography.

HEALTH HAZARDS OF X-RAYS

- Prolonged exposure destroys body tissues, damages blood cells and eye sight.
- They produce changes in subsequent generations.

DIFFERENCES BETWEEN X-RAYS AND CATHODE RAYS

X-RAYS	**CATHODE RAYS**
They are electromagnetic waves	They are fast moving electrons
They move at a speed of light	They are slower compared to x-rays
They eject electrons from matter	They produce x-rays on striking matter
They have a higher penetrating power	They have a lower penetrating power
They can't be deflected by electric or magnetic field	They are deflected by electric or magnetic fields

DIFFERENCES BETWEEN X-RAYS AND BETA PARTICLES

X-RAYS	**BETA PARTICLES**
They are electromagnetic waves	They are electrons
They have no charge	They have a negative charge
They move at a speed of light	They are slower compared to x-rays
They can't be deflected by electric or magnetic field	They are greatly deflected by electric or magnetic fields

Methods used to detect x-rays

- Photographic methods,
- Ionization chamber,
- Geiger Muller tube wit rate meter.

X-RAY DIFFRACTION

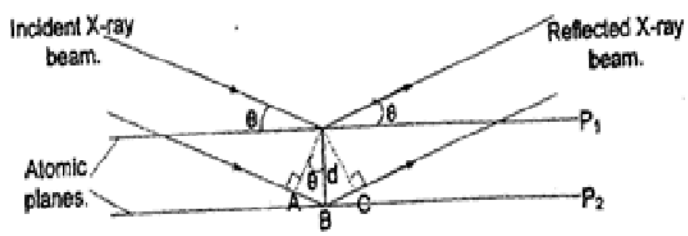
Bragg's law

It states that for adjacent crystal planes separated by a distance, d, the path difference $(2d\sin\theta_n)$ is equal to the product of the order of diffraction, n, for image formation and wavelength, λ.

That is, $2d\sin\theta_n = n\lambda$, where θ_n is the glancing angle of n^{th} order.

Derivation of Bragg's law

Consider a beam of monochromatic x-rays incident on a crystal. A small fraction of the incident x-rays is scattered by each atom in the plane. The scattered x-rays interfere constructively in those directions for which the angle of incidence is equal to the angle of reflection.(that is, scattered rays are parallel) otherwise destructive interference occurs.



For adjacent planes P_1 and P_2,

Path difference $\Delta = AB + BC$

$= d\sin\theta + d\sin\theta$

$= 2d\sin\theta$

For constructive interference, $\Delta = n\lambda$

Therefore $n\lambda = 2d\sin\theta$ which is Bragg's law for n^{th} order where,θ is the Bragg angle.

NOTE:

- Glancing angle θ is the angle between the x-ray beam and the crystal surface.
- For maximum order of diffraction n_{max} , $\sin\theta = 1$

  That is, $2d = \lambda n_{max}$
- For maximum wavelength, , λ_{max} , $n = 1$, *and* $\sin\theta = 1$

  That is, $2d = \lambda_{max}$

Atomic spacing in crystals

Consider an ionic crystal of potassium chloride $[K^+Cl^-]$

Let M = Molar mass of potassium chloride.

Mass of an ion pair $= \frac{M}{N_A}$

Let ρ density of $[K^+Cl^-]$

$$volume\ of\ an\ ion\ pair = \frac{mass}{density} = \frac{m}{\rho} = \frac{M}{N_A\rho}$$

$$volume\ of\ one\ ion = \frac{1}{2}\left(\frac{M}{N_A\rho}\right) = d^3$$

Therefore, $d = \sqrt[3]{\frac{M}{2N_A\rho}}$

WORKED EXAMPLES

1. An x-ray tube operates at 30kV and a current through it is 2mA. Calculate;
   (i) The electrical power in-put.
   (ii) The number of electrons striking the target per second.
   (iii) The speed of the electrons when they hit the target.
   (iv) The lower wavelength limit of the x-rays emitted.

Solution

$V = 30kV = 30{,}000V,\ I = 2mA = 0.002A$

(i) Electrical power in-put

$$P = IV = 0.002 \times 30000 = 60W$$

(ii) From $Q = ne$

The number of electrons striking the target per second is given by;

$$\frac{n}{t} = \frac{I}{e} = \frac{0.002}{1.6\times10^{-19}} = 1.25 \times 10^{16}s^{-1}$$

(iii) From $\frac{1}{2}m_e {v_{max}}^2 = eV$

$${v_{max}}^2 = \frac{2eV}{m_e} = \frac{2\times1.6\times10^{-19}\times30000}{9.11\times10^{-31}} = 1.054 \times 10^{16}$$

$$v_{max} = \sqrt{1.054 \times 10^{16}} = 1.027 \times 10^8 ms^{-1}$$

(iv) $\lambda_{min} = \frac{hc}{eV} = \frac{6.63\times10^{-34}\times3\times10^8}{1.6\times10^{-19}\times30000} = 4.144 \times 10^{-11}m$

2. In an x-ray tube, 90% of the electrical power supplied to the tube is dissipated as heat. If the accelerating voltage is 75kV, and the power of 742.5W is dissipated as heat, find the number of electrons arriving at the target per second. What would be the effect of increasing the voltage?

Solution

V = 75kV = 75000V

Let P be the electrical power supplied

$$power\ dissipated = 90\%\ of\ P$$

$$742.5 = 0.9P$$

$$P = \frac{742.5}{0.9} = 825W$$

Rate of flow of charge $I = \frac{P}{V} = \frac{825}{75000} = 0.011\ Cs^{-1}$

From $Q = ne = It$, *the number of electrons arriving at the target per second* is given by;

$$\frac{n}{t} = \frac{I}{e} = \frac{0.011}{1.6\times10^{-19}} = 6.875 \times 10^{16} s^{-1}$$

If the accelerating p.d is increased, kinetic energy of the electrons reaching the target increases but the number of electrons doesn't change. The heat dissipated in the target also increases.

3. The closest spacing between the planes of ions in the crystal of sodium chloride is 2.82 x 10^{-7}m. The first order reflection of monochromatic beam of reflection occurs at an angle of 15.8°. Find the wavelength of the x-rays.

Solution

$$n = 1, d = 2.82 \times 10^{-7} m\ , \theta = 15.8^0$$

$$n\lambda = 2d \sin\theta$$

$$1 \times \lambda = 2 \times 2.82 \times 10^{-7} \times \sin 15.8$$

$$\lambda = 1.536 \times 10^{-7} m$$

4. Abeam of x-rays of wavelength 1×10^{-10} m is incident on a set of cubic planes in sodium chloride (NaCl) crystal. The first order diffraction beam is obtained at glancing angle of 10.2°. Find the spacing between the consecutive planes and the density of sodium chloride. [Na = 23, Cl = 35.5]

Solution

$$n = 1,\ \lambda = 1 \times 10^{-10}\ m, \theta = 10.2^0$$

$$NaCl = Na + Cl = 23 + 35.5 = 58.5gmol^{-1}$$

$$M = \frac{58.5}{1000} = 0.0585kgmol^{-1}$$

$$n\lambda = 2d\sin\theta$$

$$d = \frac{n\lambda}{2\sin\theta} = \frac{1\times1\times10^{-10}}{2\sin 10.2} = 2.824\times10^{-10}m$$

$$volume\ of\ one\ ion = \frac{1}{2}\left(\frac{M}{N_A\rho}\right) = d^3$$

$$\rho = \frac{M}{2N_A d^3}$$

$$\rho = \frac{0.0585}{2\times6.02\times10^{23}\times(2.824\times10^{-10})^3} = 2.157\times10^3\ kgm^{-3}$$

Trial Questions

1. The deflection of a voltmeter when a p.d of 60kV is applied across an x-ray tube, a current of 30mA falls. The anode is cooled by water at a rate of 0.06ks^{-1}. If 99% of the power supplied is converted to heat at the anode, calculate the rate at which the temperature of water rises.

   [Ans: 7.07^0Cs^{-1}]

2. Electrons of energy 75keV are stopped at the target of the x- ray tube. Calculate the minimum wavelength of the x-rays produced. [Ans: $1.65\times10^{-11}m$]

3. An x-ray operates with a p.d of 100kV between the anode and the cathode. The tube current is 20MA. Calculate;

i. The rate at which energy is transformed in the target of the x-ray tube,

ii. The number of electrons which reach the target each second,

iii. The maximum energy of the x-ray photons produced.

   [Ans: $2\times10^{12}Js^{-1}, 1.6\times10^{-14}$]

4. Calculate the glancing angle of a second order diffraction of x- ray beam of wavelength $2\times10^{-11}m$ incident on a crystal with spacing between the atomic planes of $1\times10^{-10}m$

   [Ans: 11.54^0]

5. X-rays of wavelength $1\times10^{-10}m$ are diffracted from a set of planes of rubidium chloride. The first order diffraction maximum occurs at 8.8°. Calculate the inter-planar spacing.

   [Ans: $3.268\times10^{-10}m$]

CHAPTER 24: ELECTRONICS

THERMIONIC EMISSION

This is the process by which loosely attached outer electrons escape from the metal surface when heated.

Mechanism of thermionic emission

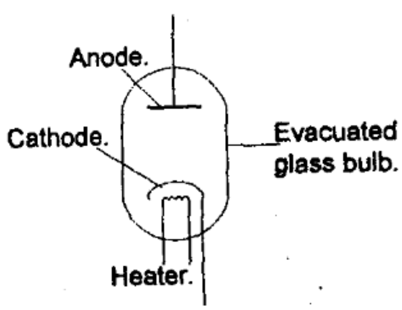
The valence/conduction electrons of a metal are loosely bound to their parent nuclei. Therefore, these electrons move randomly throughout the metal lattice.

When a metal is heated sufficiently to high temperatures, the electrons ate the metal surface gain sufficient kinetic energy to overcome their attraction by the atomic nuclei within the surface of the metal.

This ejection of electrons from the heated metal is called thermionic emission.

THERMIONIC DIODE

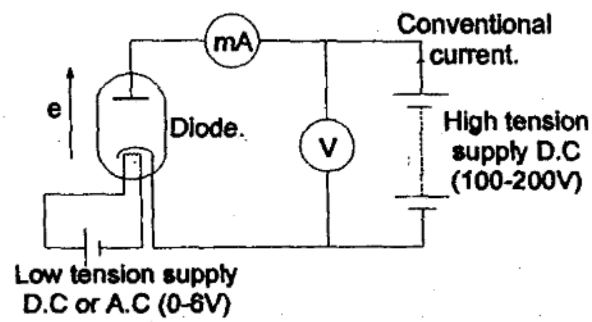
A thermionic diode is also known as a high vacuum diode. It's a device which using the principle of thermionic emission and allows current to flow in only one direction. A diode is therefore a valve since it allows current to flow in one direction.



The diagram above shows a high vacuum diode. Both the cathode and anode are mounted in the evacuated glass bulb. When the cathode is heated by a low tension supply, electrons are emitted from the supply by thermionic emission.

If the anode is maintained at the positive terminal with respect to cathode, a current flows in the circuit. This current is known as the anode current. On the other hand, if the anode is made negative with respect to the cathode due to overcrowding of electrons, cathode electrons are repelled by the anode and no current flows.

The diode circuit

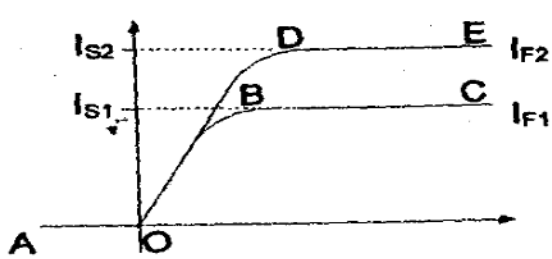


Anode current, I_a is measured by the milliammeter (mA). Anode voltage, V_a is measured by the voltmeter (V). The low tension supply maintains the heating current through the filament. The high tension supply maintains the anode at a positive potential. Low tension supply can be a.c or d.c usually between 6.3V and high tension supply must be direct (d.c) and between 100-200V; depending on the valve. The filament is joined to a high negative tension supply to provide a complete circuit for the anode current.

Control of the diode

1) **Varying the voltage of the low tension supply:** Having a high voltage implies that the cathode will have a higher temperature and hence more electron emissions.
2) **Varying the anode voltage**: Increasing the anode voltage doesn't affect the rate of emission of electrons from the cathode. It however increases the number of electrons reaching the anode.

Characteristics of a thermionic diode



I_{s1} is the filament current at filament current I_{F1} while I_{S2} is the saturation current at a higher filament current I_{F2}. Along AO, the anode is at a negative voltage relative to the cathode. All the electrons emitted by the cathode are repelled back to its surface and therefore no current flows. At O, the anode voltage is zero. The electrons, which are emitted by the filament, have small emission speeds and tend to cluster (stay) around the filament thus forming an electron cloud. The electron cloud formed is known as space charge which exerts a repulsive force on other electrons being emitted. Along OB, the anode voltage is positively increasing. Some of the outer electrons are attracted to the anode and current increases as far as point B. the current is said to be space-charge-limited along OB.

Beyond point B, the value of the anode voltage is sufficiently large such that the space charge ceases to exist and all the electrons reach the anode. Any further increase in anode voltage has very little effect on the anode current and the current is said to have reached its saturation value.

The current is said to be temperature-limited along BC. At higher filament currents, the cathode temperature is greater and more electrons are emitted, resulting in an increased saturation current as shown by the doted curve above.

EXPLANATION OF THE TERMS USED

(1) Space charge

At low voltages of the anode, few electrons are attracted. Therefore, large numbers of electrons gather together close to the cathode and exist there as an almost stationery cloud of a negative charge called the space charge. Therefore, space charge is the charge of the electron cloud.

(2) Space charge limitation

When the cathode of a vacuum diode is heated, electrons are emitted. If the anode voltage V_a is small, only those electrons emitted with high speeds will be able to reach the anode. The majority of the electrons are emitted with low kinetic energies and are repelled back towards the cathode.

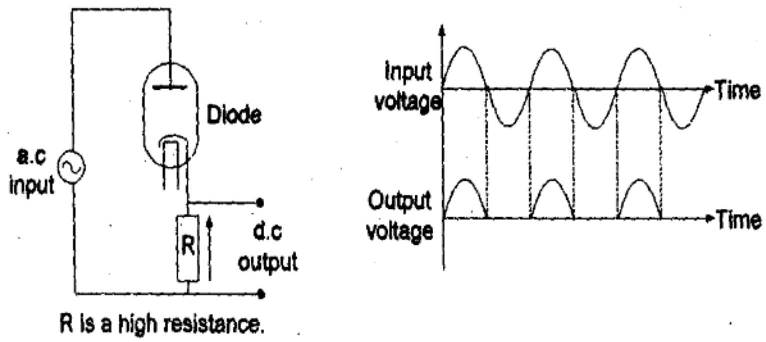
The electron distribution around the cathode constitutes the space charge. However, as anode voltage increases, the attraction of the space charge increases, anode current increases and space charge limitation is said to have occurred.

(3) Saturation

When the anode voltage is so large, all the electrons emitted per second by the cathode reach the anode. Space charge is overcome and current becomes constant (saturation current). At this stage, Saturation is said to have occurred.

Diode as rectifier

Rectification is the process of converting a.c to d.c for use in radios, televisions e.t.c. a diode is used for this purpose because it permits electrons to flow in one direction. The figure below shows the circuit of a diode as a rectifier.



During the positive half cycle of the a.c input waveform, the anode (A) is made positive with respect to the cathode (K). The anode collects all the electrons from the cathode and current flows in the anode circuit. The output voltage across the load R is in the direction as shown above.

During the negative half cycle of the a.c input waveform, the anode (A) is made negative with respect to the cathode (K). Electros from the cathode are repelled by the anode and no current flows in the anode circuit. Then the output voltage across R is zero. Hence, the output consists of a series of half waves in one direction and half-wave rectification is produced.

RECTIFICATION

Rectification is the conversion of alternating current or voltage into direct current or voltage. The rectified current or voltage flows in one direction and has a constant amplitude.

A rectifier circuit is a circuit that converts an a.c supply into a pulsating a.c supply. There are two types of rectifier circuits namely: half-wave rectifier circuit and full-wave rectifier circuit.

(1) Half-wave rectification

The figure below shows a half-wave rectifier circuit. It consists of a diode, D_1 in series with the a.c input to be rectified and the load, R requiring the d.c output.

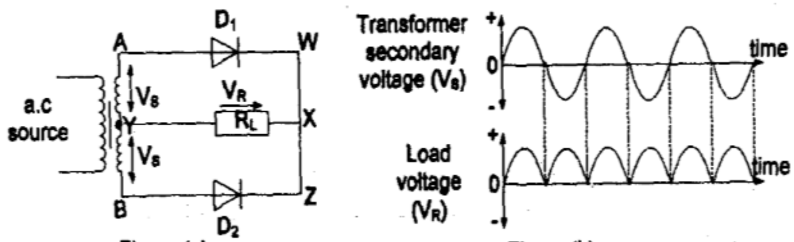


Figure (a) Figure (b)

During the positive half cycle of the a.c input waveform, point A is made positive with respect to point B. Diode D_1 conducts and a current flows in the load R. During the negative half cycle, diode D_1 is non-conducting and no current flows.

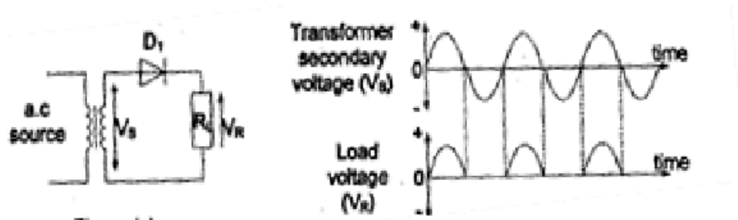
NOTE: The major disadvantage of this simple rectifying circuit is that the load voltage varies considerably and is zero for half the time. Such a waveform is only suitable for simple applications such as battery charging.

Full-wave rectification

In this process, both halves of every cycle of input voltage produce current pulses. There are two types of full wave rectifier circuits namely centre-tap and bridge rectifier.

Centre-tap full-wave rectifier

The figure below shows a centre-tap full-wave rectifier circuit.



The secondary winding of the input transformer is accurately centre-tapped so that equal voltages are applied across the two diodes D_1 and D_2. Centre-tap is a point that is situated in the middle of the number of secondary winding.

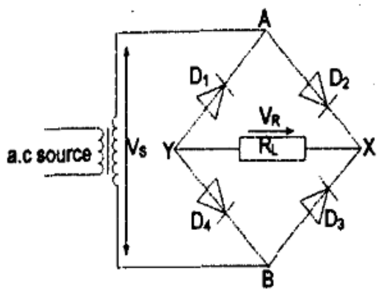
During the positive half cycle of the a.c input waveform, point A is made positive with respect to point Y while point B is made negative with respect to point Y. Diode D_1 therefore conducts while D_2 doesn't. This gives a current pulse in the direction *AWXY* thus current flows through the load R in the direction XY. During the negative half cycle of the a.c input waveform, point A is made negative with respect to point Y while point B is made positive with respect to point

Y. Diode D_2 therefore conducts while D_1 doesn't. This gives a current pulse in the direction *BZXY* thus current flows through the load R in the same direction *XY* as before.

NOTE: The disadvantage of this circuit is the need of a centre- tapped transformer and for two diodes.

Bridge rectifier circuit

The bridge rectifier requires four diodes but eliminates the need of a centre-tapped input transformer.



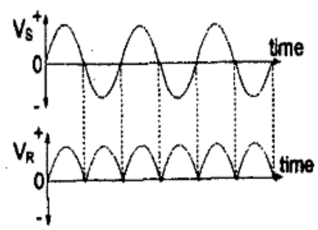


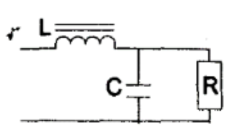
Figure (b)

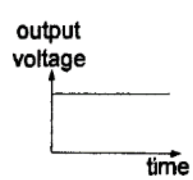
During the positive half cycle of the a.c input waveform, point A is made positive with respect to point B. Diodes, D_2 and D_4 are conducting while diodes D_1 and D_3 are non-conducting. This gives a current pulse in the direction *AXYB* thus current flows through the load R in the direction *XY.*

During the negative half cycle of the a.c input waveform, point B is made positive with respect to point A. Diodes, D_1 and D_3 are conducting while diodes D_2 and D_4 are non-conducting. This gives a current pulse in the direction *BXYA* thus current flows through the load R in the same direction *XY as* before.

Filter circuits

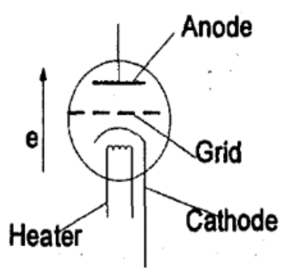
The output from half or full wave rectifier circuit is not a steady voltage like that from a battery. It would not be good to drive an amplifier or radio receiver since ups and downs would be amplified and give an uncomfortable hum in the loudspeaker. Filter circuits are therefore used to smoothen rectified output voltage.





In rectification of using a diode, varying voltages are realized. If a filter circuit is used, the time-variation in the rectifier causes an e.m.f to be induced in the inductor of inductance L in such a direction so as to oppose the variation. This results into a voltage which is a little bit steady. The remaining time-variation in voltage is taken up by the capacitor of capacitance C which is arranged in parallel with a resistor R. the capacitor becomes charged up after the level of the peak output voltage and this enables it to supply current even when the voltage falls. The voltage across R is therefore steady (smooth).

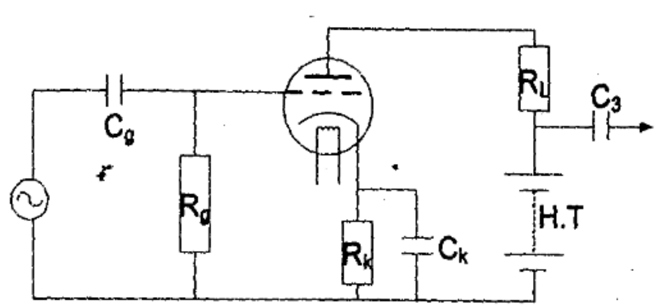
TRIODE VALVE



A triode has three electrodes namely:

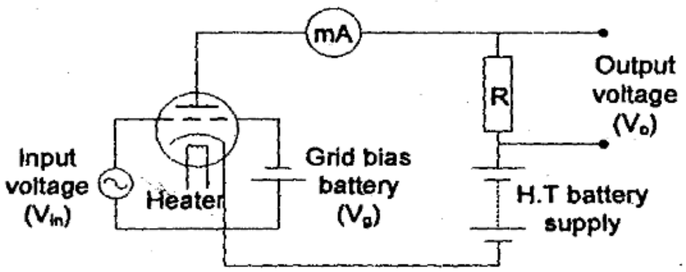
1) The node.
2) The grid (spiral wire) wound close to cathode but does not touch the cathode so that electrons can pass through the spaces between its windings.
3) The cathode.

A single-stage triode amplifier circuit



The function of capacitance C_g is to block the d.c component of the grid bias and to allow only the a.c component to flow. The function of the capacitance C_k is to make the grid bias voltage steady. Capacitance C_3 separates an alternating p.d from a direct p.d at the output of the triode. Resistance R_g provides the return path for electrons on the grid. The output of the amplifier is taken across the load resistance R_L.

Using a triode as a voltage amplifier



The diagram above shows how a triode can be used as a voltage amplifier. The grid must always be negative with respect to the anode.

The input alternating voltage V_{in} ,to be amplified, causes alternating variations/changes in the grid bias voltage V_g. If the grid is positive with respect to the cathode, more electrons pass and this causes the grid current to flow. By definition,

$$\begin{pmatrix} amplification \\ factor, \mu \end{pmatrix} = \frac{change\ in\ anode\ voltsge}{change\ in\ grid\ potential} = \frac{\Delta V_a}{\Delta V_g}$$

Thus $\Delta V_a = \mu \Delta V_g$

If the grid is close to the cathode, a small change of grid potential causes an appreciable change (i.e a large variation) of anode current, I_a. From $V_0 = I_a R$ the large variation in the anode current implies that the output voltage, V_0 across the load R will be much larger than V_{in}.

The triode valve acts as a voltage generator of $e.m.f = \mu V_{in}$. If R_a is the resistance of the anode (internal resistance of the triode) in series with the load resistance R,

$$\begin{pmatrix} current\ through \\ the\ load \end{pmatrix} = \left[\frac{e.m.f}{total\ resistance}\right]$$

$$I_a = \left[\frac{\mu V_{in}}{R_a + R}\right]$$

Output voltage, $V_0 = I_a R = \left[\frac{\mu V_{in}}{R_a + R}\right] R$

By definition, voltage gain, $A_v = \frac{output\ voltage, V_0}{input\ voltage, V_{in}}$

$$A_v = \frac{V_0}{V_{in}} = \frac{1}{V_{in}} \times \left[\frac{\mu V_{in}}{R_a + R}\right] R = \frac{\mu R}{R_a + R}$$

NOTE:

1)The waveform of V_0 is the same as that of V_{in} since the voltage gain in independent of the frequency of the alternating voltage.

2) The output current has A.C and D.C components. However if the triode valve is being used as a voltage amplifier, the output voltage V_0 concerned is that developed across the load resistance R by the A.C component only.

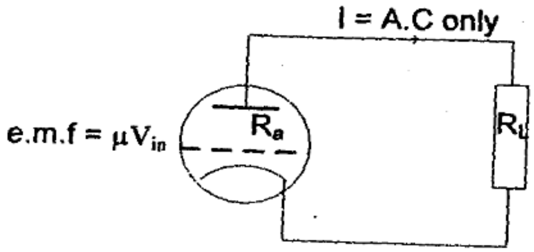
3) The A.C voltage is separated from the D.C voltage in the output by using a capacitor which is

connected along the path of current.

4) Amplification factor, μ can also be expressed as

$$\begin{pmatrix} amplification \\ factor \end{pmatrix} = \begin{pmatrix} anode \\ resistance \end{pmatrix} \times \begin{pmatrix} mutual \\ conductance \end{pmatrix}$$

An equivalent circuit of a triode as an amplifier



WORKED EXAMPLES

1. A triode with mutual conductance of 4.0mV^{-1} and an anode resistance of 5kΩ is connected to a load resistance of 10kΩ.Assuming that the triode is operating under optimum conditions, estimate the output signal obtained for an alternating input of 25mV.

Solution

$V_{in} = 25 \times 10^{-3}V,\ R_a = 5000\Omega\ , R = 10000\Omega, g_m = 4 \times 10^{-1}AV^{-1}$

$$\begin{pmatrix} amplification \\ factor \end{pmatrix} = \begin{pmatrix} anode \\ resistance \end{pmatrix} \times \begin{pmatrix} mutual \\ conductance \end{pmatrix}$$

$\mu = R_a \times g_m = 5000 \times 4 \times 10^{-3} = 20$

Output voltage $V_0 = \left[\frac{\mu V_{in}}{R_a + R}\right] R$

$V_0 = \left[\frac{20 \times 25 \times 10^{-3}}{5000 + 10000}\right] \times 10000 = 0.333V$

2. A sinusoidal voltage of amplitude 0.2V is applied to the grid of a triode of amplification factor 10. If the anode resistance of the triode is 15kΩ, what voltage will appear across a grid of 10kΩ?

Solution

$V_{in} = 0.2V, \mu = 10\ , R_a = 5000\Omega, R = 10000\Omega$

Output voltage $V_0 = \left[\frac{\mu V_{in}}{R_a + R}\right] R$

$$V_0 = \left[\frac{10\times0.2}{5000+10000}\right] \times 10000 = 1.333V$$

SEMICONDUCTORS

A semiconductor is a substance whose resistivity is much less than that of an insulator but much greater than that of a conductor; and whose resistivity decreases with increase in temperature. For example germanium, silicon, carbon e.t.c.

Properties of semiconductors

1) The resistivity of a semiconductor is less than that of an insulator but more than that of a conductor.
2) Semiconductors have a negative temperature coefficient of resistivity, i.e. resistance of a semiconductor decreases with increase in temperature.
3) When a suitable metallic impurity is added to a semiconductor, its current conduction properties change appreciably.

CLASSIFICATION OF SEMICONDUCTORS

(1) Intrinsic and extrinsic semiconductors

A semiconductor in an extremely pure form is known as an intrinsic semiconductor. The pure semiconductor has little current conduction capability at room temperature. To be useful in electronic devices, the pure semiconductor must be altered so as to significantly increase its conducting properties. This is achieved by adding a small amount of suitable impurity to a semiconductor. It's then called an impurity/extrinsic semiconductor. The process of adding impurities to a semiconductor is known as **doping**. The purpose of doping is to increase either the number of free electrons or holes in the semiconductor crystal.

Thus, by definition, **an intrinsic semiconductor** is a semiconductor whose crystal is not doped by any impurity atom. On the other hand, an **extrinsic semiconductor** is a semiconductor whose crystal has been doped with some small amount of impurity atoms.

(2) n-type and p-type semiconductors

(i). n-type semiconductor

When a small amount of pentavalent impurity (known as a donor atom) is added to a pure semiconductor, the resultant is known as an n-type semiconductor. A donor atom is defined as a pentavalent impurity atom that donates a free electron to the crystal lattice. Examples of donor atoms include: aluminium, boron, indium and gallium. Addition of a pentavalent impurity (donor) provides a large number of free electrons in the semiconductor. Therefore, current conduction in n-type semiconductor is predominantly by free electrons. Thus, in an n-type semiconductor, free electrons are the majority charge carrier while holes are the minority charge carriers

(ii) p-type semiconductor

When a small amount of trivalent impurity (known as a acceptor atom) is added to a pure semiconductor, the resultant is known as a p-type semiconductor. An acceptor atom is defined as a trivalent impurity atom which gives rise to creation of a hole (positive large) in the crystal lattice. Examples of donor atoms include: phosphorus, antimony and arsenic. Addition of a trivalent impurity (acceptor) provides a large number of holes in the semiconductor. Therefore, current conduction in a p-type semiconductor is predominantly by holes. That is, in a p-type semiconductor, holes are the majority charge carrier while, free electrons are the minority charge carriers.

SEMICONDUCTOR DIODE {P-N JUNCTION DIODE}

A single crystal of silicon or germanium, which has been doped in such a way that one half of it is p-type and the other is n-type, can be used as a rectifier. It's the existence of a junction between the *two* types which gives the device its ability to rectify; it's therefore called a p-n junction diode.

The semiconductor diode as a rectifier

A semiconductor diode is a device that has *a* low resistance to the f low of current in one direction and a high resistance in the other. That is a semiconductor diode allows the electric flow in one direction only and therefore can be used as a rectifier. The circuit symbol of a p-n junction diode is as shown in figure (b) below.

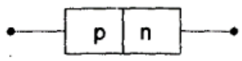


Figure (a)

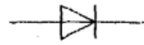


Figure (b)

Current - Voltage characteristics of a semiconductor diode

The current - voltage characteristic of a semiconductor diode is a graph that shows the relationship between the current flowing in the device and the voltage applied across it. The circuit in fig (a) below is used to get the forward bias characteristics and that in fig (b) is used to obtain the reverse bias characteristics

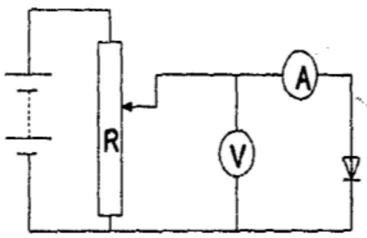


Fig (a). Forward bias circuit.

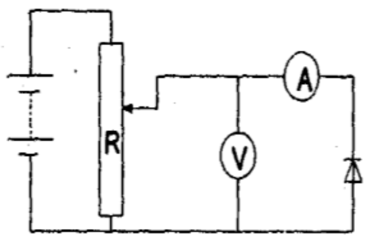
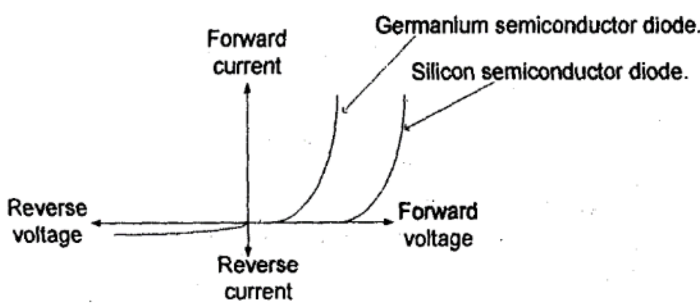


Fig (b). Reverse bias circuit.

The applied voltage is increased from zero in a number of steps and the current flowing in each step noted. The noted current values are then plotted on the corresponding values of voltages. The current - voltage characteristic of a semiconductor diode are as follows.



The a.c resistance of the diode at a particular d.c voltage is equal to the reciprocal of the slope of the characteristic at that point. That is

$$R_{a.c} = \frac{change\ in\ voltage}{change\ in\ current} = \frac{\Delta V}{\Delta I}\,(\Omega)$$

The d.c resistance of the diode at a particular point of the characteristic is the ratio of the applied voltage to the current flowing across it. Since the characteristic of the diode is non-linear, the d.c resistance will vary with the point of measurement.

TRANSISTORS

A transistor consists of two p-n junctions formed by sandwiching either p-type or n-type semiconductor between a pair of opposite types. One junction is forward biased and the other is reversed biased. The forward biased junction has a low resistance whereas the reverse biased junction has a high resistance path.

Types of transistors

There are two types of transistors, namely: n-p-n transistor and p-n-p transistor. An n-p-n transistor is composed of two n-type semiconductors separated by a thin section of p-type as shown in figure (a) below. However, a p-n-p transistor is formed by two p- sections separated by a thin section of n-type as shown in figure (b) below.

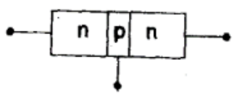


Figure (a)

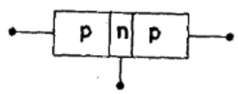


Figure (b)

NOTE: In each type of transistor, the following points may be noted:

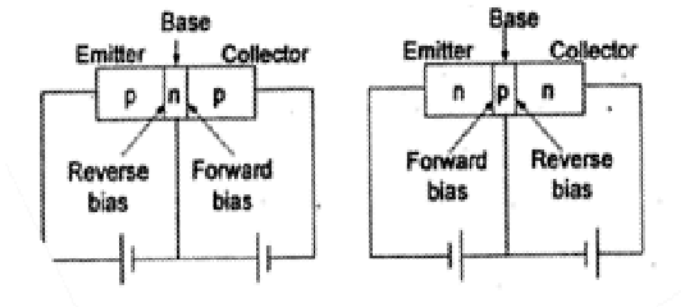
- There are two p-n junctions. Therefore a transistor may be regarded as a combination of two diodes connected back to back.
- There are three terminals taken from each type of semiconductor.
- The middle section is a very thin layer. This I the most important factor in the function of a transistor.

Naming of transistor terminals

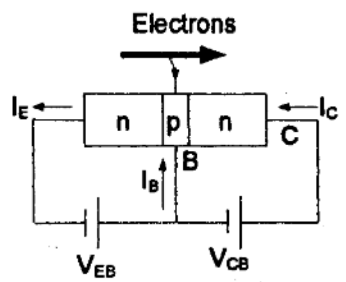
A transistor (p-n-p or n-p-n) has three sections of doped semiconductors. The section on one side is the emitter and the section on the other side is the collector. The middle section is called the base and forms two junctions between the emitter and collector.

1) **Emitter:** The section on one side that supplies charge carriers (electrons or holes) is called the emitter. The emitter is always forward biased with respect to the base so that it can supply a large number of majority carriers.
2) **Collector**: The section on the other side that collects the charges is called the collector. The collector is always reverse biased its function is to remove charges from its junction with the base.
3) **Base**: The middle section which forms two p-n junctions between the emitter and collector is called the base. The base- emitter junction is forward biased, allowing low resistance for the

emitter circuit. The base-collector junction is reverse biased and provides high resistance in the collector circuit

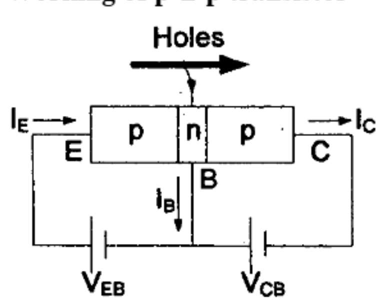


Working of n-p-n transistor



The figure above shows the n-p-n transistor with forward bias to emitter-base junction and reverse bias to collector-base junction. The forward bias causes the electrons in the n-type emitter to flow towards the base. This constitutes the emitter current I_E. As these electrons flow through the p-type base, they tend to combine with the holes. As the base is lightly doped and very thin, therefore, only a few electrons (less than 5%) combine with holes to constitute base current I_B. The remainder (more than 95%) cross over into the collector region to constitute collector current I_c. In this way, almost the entire emitter current flows into the collector circuit. It's clear that emitter current is the sum of collector and base currents i.e. $I_E = I_B + I_C$.

Working of p-n-p transistor

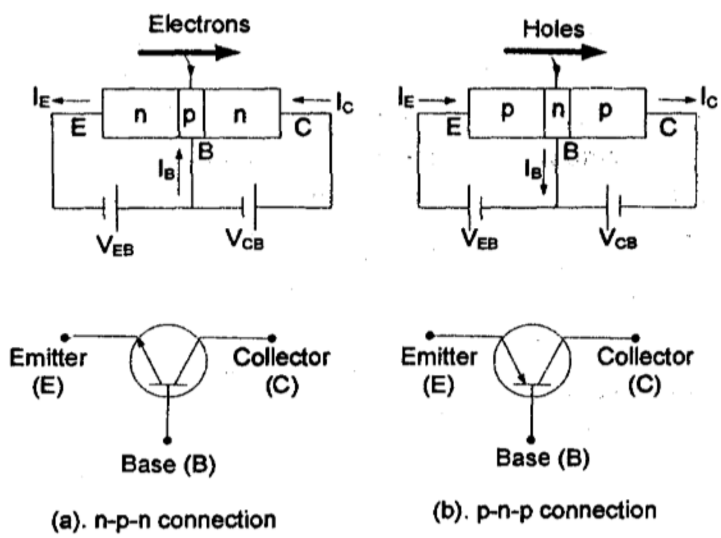


The figure above shows the basic connection of a p-n-p transistor. The forward bias causes the holes in the p-type emitter to flow towards the base. This constitutes the emitter current I_E. As these holes cross into the n-type base, they tend to combine with the electrons. As the base is lightly doped and very thin, therefore, only a few holes (less than 5%) combine with electrons to constitute base current I_B. The remainder (more than 95%) cross over into the collector region to constitute collector current I_c. In this way, almost the entire emitter current flows into the collector circuit. It's clear that emitter current is the sum of collector and base currents i.e $I_E = I_B + I_C$

It may be noted that current conduction within p-n-p transistor is by holes. However, in the external connecting wires, the current is still by electrons.

Transistor symbol

In the earlier diagrams, the transistors have been shown in diagrammatic form. However, for the sake of convenience, the transistors are represented by schematic diagrams. The symbols used for n-p-n and p-n-p transistors are shown in figure below.



Note that emitter is shown by an arrow which indicates the direction of conventional current flow with forward bias. For n-p-n connection, it's clear that conventional current flows out of the emitter as indicated by the outgoing arrow in figure (a). Similarly, for p-n-p connection, the conventional current flows into the emitter as indicated by the inward arrow in figure (b)

Advantages of a transistor as an amplifier over a triode

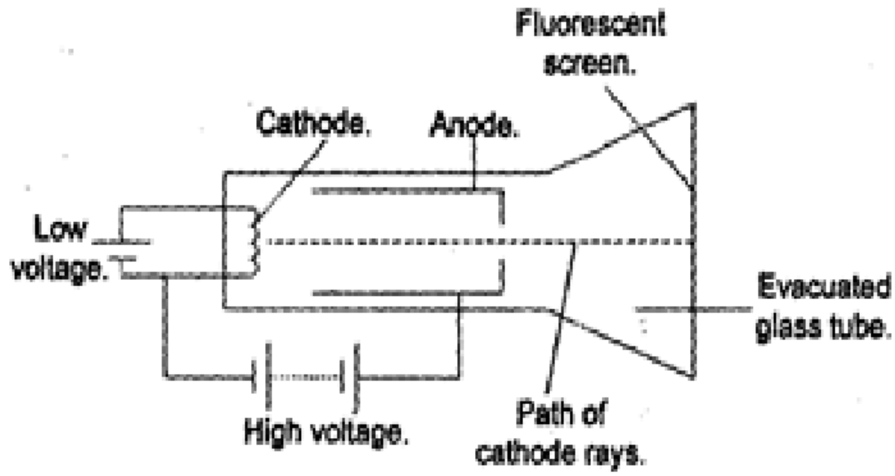
Triode	**Transistor**
Needs high voltage	Needs low voltage
A cathode heater is required to produce electrons	No heater is required to produce electrons. Current carriers are just available in the semiconductor
Has a vacuum which can deteriorate i.e goes soft when a gas is introduced into it. So, unnecessary ionization occurs and the cathode becomes poisoned	Has no vacuum and therefore no deterioration if used correctly. Has a short life span provided the power rating is not exceeded.

CHAPTER 25: ATOMIC PHYSICS

CATHODE RAYS

Cathode rays are a stream of fast moving electrons.

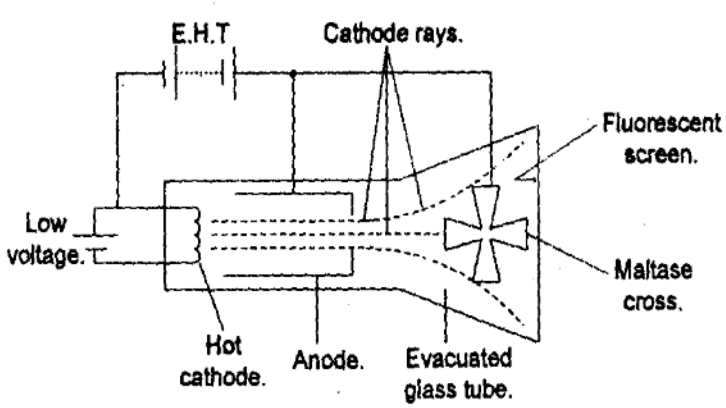
Production of cathode rays



Electrons are emitted by thermionic emission when the cathode is heated by the low voltage supply. Emitted electrons are then accelerated through a p.d of several kilovolts between the cathode and the anode.

When they reach the anode, some of them pass through the anode and hit the screen coated with a fluorescence material which causes the screen to glow. It's this beam of fast moving electrons from the cathode which constitutes of the cathode rays.

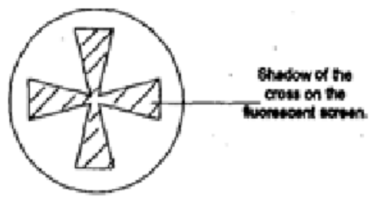
Experiment to show that cathode rays move in a straight line



The set up above consists of a hot cathode and a cylindrical anode in an evacuated glass tube/ envelope;

having a coating of a fluorescent material inside it. The anode and the maltase cross are connected to the positive terminal of the extra high tension (E.H.T) so that the electrons re-accelerate along the tube in a divergent beam.

Observations show that the electrons by pass the anode and a dark shadow of the cross appears on the screen against a colored fluorescent background.



This shows that the rays are moving in straight lines from the cathode and those not intercepted by the cross cause the screen to fluorescent

Determining the sign of charge of cathode rays

When the electron beam is made to pass through an electric field between two plates, it's seen to deflect towards the positively charged plate implying that it's negatively charged.

Properties of cathode rays

i. They travel in straight lines.
ii. They cause fluorescence in some substances like glass,
iii. They can be deflected by electric and magnetic fields,
iv. When they strike objects, heat is produced.
v. X-rays are produced when they strike matter.

ELECTRON DYNAMICS

If cathode rays are assumed to consist of particles (electrons) to which the laws of mechanics apply, we can obtain information about their speed and specific charge from their behavior in electric and magnetic fields.

(a) Speed of electrons

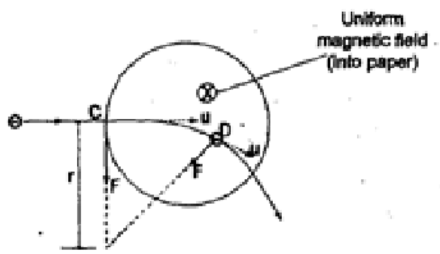
Consider an electron of charge e and mass m_e which is emitted from a hot cathode and then accelerated by an electric field towards an anode. It experiences a force due to the field and work is done on it. The system (of field and electron) loses electrical potential energy and the electron gains kinetic energy. Let V_{ak} be the p.d between the anode and cathode; responsible for the field. If the electron starts from the cathode with zero speed and moves in a vacuum attaining speed v as it reaches the anode,

$$(electrical\ work\ done) = (kinetic\ energy\ gained\ per\ electron)$$

$$eV_{ak} = \frac{1}{2}m_e v^2$$

$$v = \sqrt{\left(\frac{2eV_{ak}}{m_e}\right)}$$

(b) Path of an electron in a magnetic field



On entering the magnetic field of flux density B, the electrons experience a magnetic force F = Beu at right angles to their direction of motion and to the field.

At any other point 'D' in the magnetic field, the magnitude of force acting on an electron is the same as at 'C since none of B, e and u has changed in magnitude but the direction of the force is different. This implies that the force only changes the direction of the electron thus causing it to move in a circular path of radius r. The constant radial force F is the centripetal force and so

$$Beu = \frac{mu^2}{r}$$

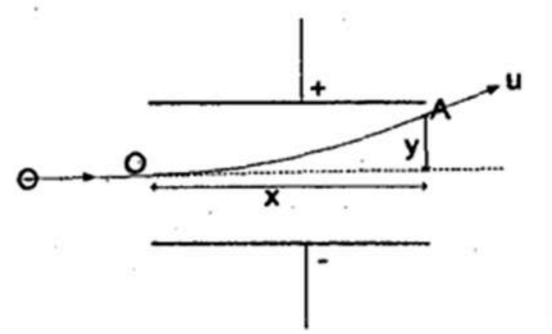
(c) Path of electrons in an electric field

On entering the electric field, the electrons experience an electric force $F = Ee$ in the direction opposite to that of the electric field. From Newton's second law,

(electric force) = (electron mass) (acceleration of the electron)

$$Ee = m_e a$$

$$a = \frac{Ee}{m_e} \rightarrow (i)$$



Considering horizontal motion,

(time taken to move from O to A) = $\frac{distance}{velocity}$

$$t = \frac{x}{u} \rightarrow (ii)$$

For vertical motion, $s = \frac{1}{2}at^2$

$$y = \frac{1}{2}\left(\frac{Ee}{m_e}\right)\left(\frac{x}{u}\right)^2$$

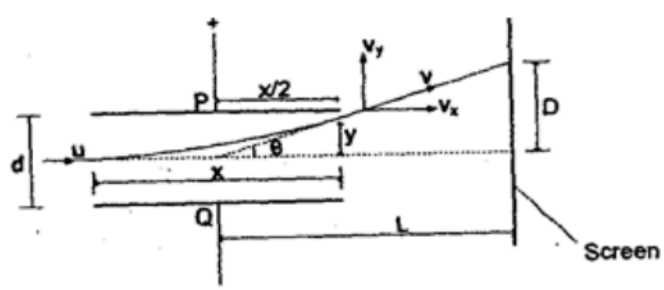
$$y = \left(\frac{Ee}{2m_e u^2}\right)x^2$$

This equation is in the form $y = kx^2, where\ k = \left(\frac{Ee}{2m_e u^2}\right)$

Hence, the path taken by an electron in a magnetic field is parabolic.

(d) Deflection of electrons in an electric field

The deflection D of the electron (its displacement from the original direction) on a screen a distance L from the centre of the plates can be obtained using the fact that it continues in a straight line after leaving the field.



From the diagram above,

P and Q are the deflecting plates. v_x and v_y are the horizontal and vertical components of velocity v of the electron respectively after passing through the deflecting plates, u is the speed of the electron on entering the electric field.

Let V_{ak} be the anode-to-cathode voltage (also known as the accelerating p.d), V be the p.d across the plates P and Q; and E be the electric field intensity between the plates. Electric field intensity is given by;

$$E = \frac{V}{d} \rightarrow (i)$$

Electric force is given by $F_e = Ee = ma$

$$a = \frac{Ee}{m}$$

Therefore, (acceleration of the electron) = (electric field intensity)× (specific charge)

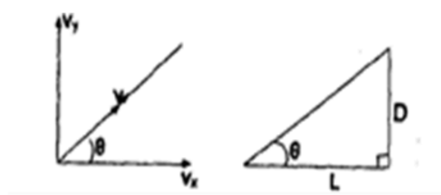
$$a = E\left(\frac{Ee}{m}\right) \rightarrow (ii)$$

Time taken by the electron to pass through the plates

$$t = \frac{x}{u} \rightarrow (iii)$$

Vertical component of velocity $V_y = at = E\left(\frac{e}{m}\right)\left(\frac{x}{u}\right) \rightarrow (iv)$

Horizontal component of velocity , $V_x = u \dots\dots (v)$

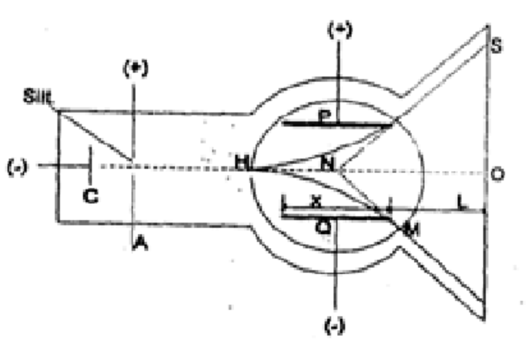


$$tan\theta = \frac{V_y}{V_x} = \frac{D}{L}$$

$$\frac{E\left(\frac{e}{m}\right)\left(\frac{x}{u}\right)}{u} = \frac{D}{L}$$

Therefore, the deflection is given by $D = \left(\frac{e}{m}\right)\left(\frac{x}{u^2}\right)EL \rightarrow (vi)$

THOMPSON'S METHOD TO DETERMINE THE SPECIFIC CHARGE OF AN ELECTRON



C is the cathode and A is the anode with a slit to a narrow beam to strike the fluorescent screen. P

and Q are parallel plates which can be connected to the battery. Electrons are emitted by the cathode C and then accelerated to the anode by the p.d between the anode and the cathode.
If the electric field is switched on by connecting the plates P and Q to the battery, the cathode rays are deflected upwards and strike the screen at S. the deflection, $D = OS$ is noted.
A magnetic field is then applied at right angles to the electric field and electric field between the plates varied until the spot is brought back to position "0".

When the spot is brought back to "0", it implies that there's no deflection of the cathode rays. Thus,

$$\text{(Magnetic force)} = \text{(electric force)}$$

$$Beu = Ee$$

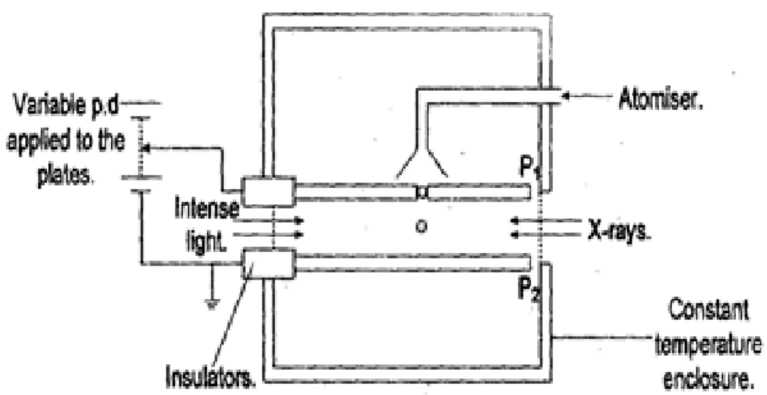
$$u = \frac{E}{B}$$

When there's no magnetic field, the deflection

$$D = \left(\frac{e}{m}\right)\left(\frac{x}{u^2}\right)EL$$

Therefore, the specific charge of an electron is given by $\left(\frac{e}{m}\right) = \frac{Du^2}{xEL}$

MILKAN'S OIL DROP EXPERIMENT



The set up is as shown above. P_1 and P_2 are metal plates separated by a distance d. P_1 has a small hole through it.
An atomizer is used to create a fine mist of oil drops through a small hole in plate P_1. The oil drops acquire charge by exposure to x-rays. The chamber is illuminated by intense light and the oil drops observed through a short focus travel microscope. One drop is selected and observed through a travel microscope when there's no p.d between the plates P_1 and P_2. Its terminal velocity v_0 is

determined by measuring the distance it falls through a measured time. A p.d is applied between the plates and it's adjusted till the drop of the oil remains stationary.

Let *p* be the density of the oil drop, σ be the density of air, *E* be the electric field strength, η be the coefficient of viscosity and *g* be the acceleration due to gravity.

When there's no p.d and the oil drop is stationery,

$$U + F = w$$

$$\frac{4}{3}\pi r^3 \rho g + 6\pi\eta r v_0 = \frac{4}{3}\pi r^3 \sigma g$$

$$6\pi\eta r v_0 = \frac{4}{3}\pi r^3(\rho - \sigma)g$$

$$r^2 = \frac{9\eta v_0}{2g(\rho - \sigma)} \rightarrow (i)$$

When there is a p.d and the oil drop is stationery,

$$U + F_e = w$$

$$\frac{4}{3}\pi r^3 \rho g + EQ = \frac{4}{3}\pi r^3 \sigma g$$

$$EQ = \frac{4}{3}\pi r^3(\rho - \sigma)g \rightarrow (ii)$$

If $\rho, \sigma, g, \eta,$ *and* v_0 are known, the radius of the oil drop can be calculated from (i). Also, if *E* is further known, the charge *Q* on the oil drop can be calculated from (ii). The procedure is repeated and the highest common multiple of *Q,* in the results, is used to determine the number of electron charges using the equation below $n = \frac{Q}{e}$

Precautions

i, A non-volatile oil should be used to avoid or to prevent evaporation which would change the mass of the oil drops.

ii. A constant temperature enclosure is used to prevent convection currents.

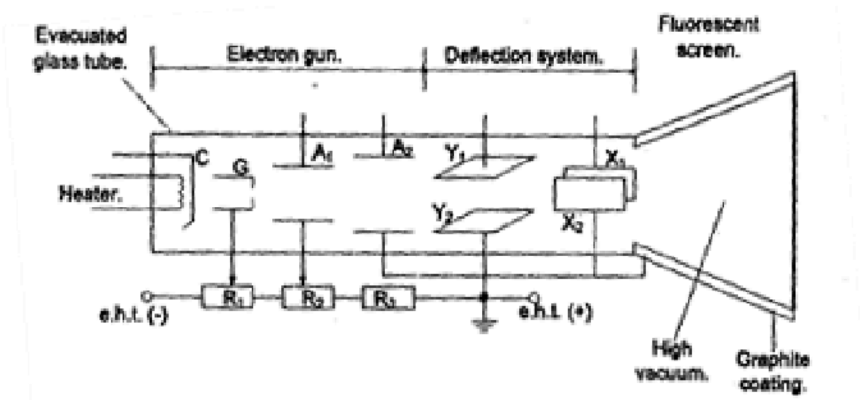
Theory of the experiment

i. Stoke's law holds.

ii. Oil drops are identical.

iii. The temperature is constant.

THE CATHODE RAY OSCILLOSCOPE (CRO)



C is the Cathode, G is the Grid control, A_1 and A_2 are anodes, Y_1 and Y_2 are y-plates (connected to the signal), X_1 and X_2 are x-plates(connected to the time base).

The cathode emits electrons by thermionic emission.

The grid is held at variable negative potential by means of the potential divider R_3.

The grid acts as the brightness control.

The anode A_1 focuses the electron beam by means of the potential divider R_2. Anode A_2 accelerates the electron beam across a highly evacuated tube.

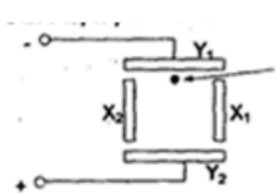
The high vacuum prevents secondary emission.

Graphite prevents the return path of electrons to the circuit to prevent accumulation of the electrons.

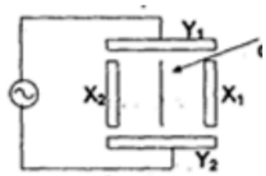
Y-plates deflect the beam vertically while X-plates deflect the beam horizontally.

The fluorescent screen is coated with zinc sulphide and therefore glows when hit by cathode rays. The display wave form is seen on the screen.

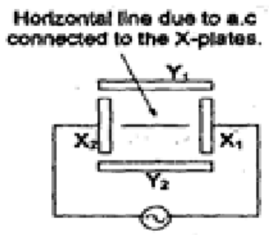
Case 1. Display of waveform if there's no signal on the X-plates.



Case 2: Display of waveform if there's alternating current (a.c) across the Y-plate and no signal on the X-plate.

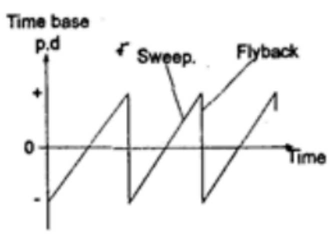


Case 3: Display of waveform if there's alternating current (a.c) connected across X-plates and no signal on the Y-plates.

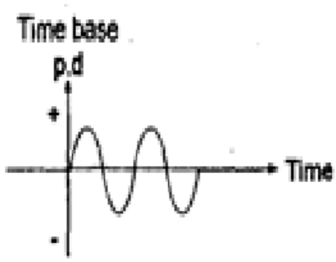


Time base

(i). Alternating current across X-plates to generate saw tooth voltage which sweeps electron beam from left to right at a constant speed.



(ii). Alternating current signal across Y-plates combined with signal on X-plate.

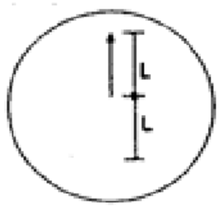


Brightness of the spot beam

Brightness is controlled by varying the p.d between the cathode and the grid. If it's made more positive, then the number of electrons passing through it per unit time will increase hence more electron will bombard the screen per unit time; making the intensity of the energy emitted, in form of light, higher. The spot therefore becomes brighter.

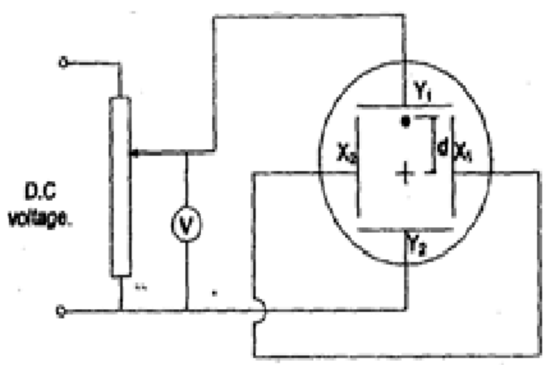
Y -sensitivity of a cathode ray oscilloscope

Consider an alternating voltage connected to the Y-plates. When the time base is switched off, the waveform displayed is as shown below.



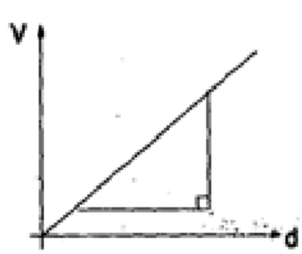
Y-sensitivity = $\frac{V_0}{L}$ where V_0 is the peak voltage measured in volts and L is the peak height; measured in centimeters.

Calibration [determining sensitivity] of a cathode ray oscilloscope



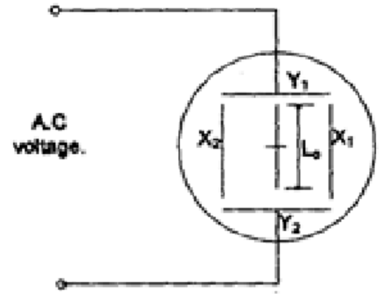
The p.d, V, measured using a voltmeter is applied across the Y-plates and varied using a potential divider. For various values of d.c voltage V, the deflection d, of the electron spot on the screen is measured using the graduated scale.

The results are tabulated and a graph of V against d plotted. The slope S is calculated and it gives the value of Y-sensitivity/ voltage gain.



Slope = [sensitivity of the C.R.O]

Measurement of a.c voltage using a cathode ray oscilloscope



An unknown a.c voltage is applied across the Y-plates. If the time base is switched off, a vertical line is obtained on the screen. This can be centered and its length L_0 measured. The peak to peak value of voltage is then equal to the product of the peak to peak height and Y-sensitivity of the C.R.O.

(peak to peak value of voltage) = (peak to peak length, L_0) (Y-sensitivity)

(peak a.c voltage) $= \frac{1}{2}$ $(peak\ to\ peak\ value\ of\ voltge)$

Uses of a cathode ray oscilloscope

It's used in:

i. Measurement of peak value of a.c and d.c voltage.

ii. Measurement of phase voltage between two a.c voltages.

iii. Measurement and comparison of frequencies.

iv. Measurement of time interval.

v. Study of waveforms in various electrical forms in various electrical circuits.

vi. It's also used as a computer output device.

Advantages of a cathode ray oscilloscope over a voltmeter

'i. It can't be damaged by overloading.

ii. It can be used both for a.c and d.c supplies.

iii. It doesn't draw any current from the circuit whose p.d is to be measured; thus giving more accurate values of p.d.

iv. Since the electron beam acts as a pointer, it has negligible inertia and therefore deflects almost instantaneously.

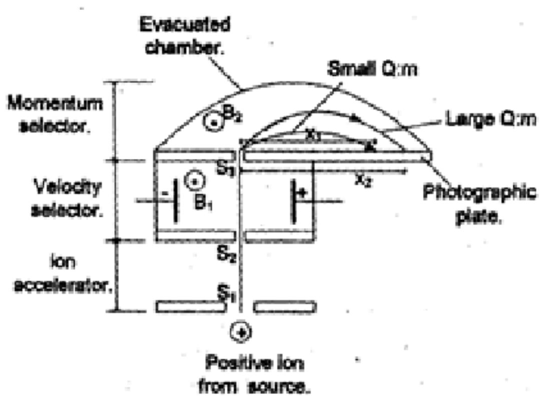
POSITIVE RAYS

Positive rays are a stream of positive ions produced by passing electrons from a hot cathode into a gas or stream of vapour.

Differences between cathode rays and positive rays

CATHODE RAYS	POSITIVE RAYS
They have a negative charge.	They have a positive charge
They move at a very high speed.	They move at a lower speed
They have the same mass.	Their mass depends on their source
They have a small mass	They have a large mass
they are less ionizing	They are more ionizing
they have a constant specific charge	Their specific charge depends on the source
They slightly affect photographic plates	They don't affect photographic plates
They are deflected to a greater extent because of their smaller mass	They are deflected to a smaller extent because they are massive.

Determination of charge to mass ratio of a positive ion using a mass spectrometer



A stream of positive ions from the ion source enter the velocity selector through the collimator slits S: and S_2. In the region between S and S, crossed uniform electric and magnetic fields are applied and at equilibrium,

[Magnetic force] = [electric force]

$$B_1Qv = EQ$$

$$v = \left[\frac{E}{B_1}\right] \rightarrow (i)$$

V is the velocity with which all the ions emerge from the slit S_3.

Therefore, only those with a velocity V =[$\frac{E}{B_1}$] pass though the velocity selector undeflected; so as to enter the momentum selector through the slit S_3.

In the momentum selector, only a uniform magnetic field is applied and the ions therefore describe a circular path of radius r, to the photographic plate. At equilibrium,

[Magnetic force] = [Centripetal force]

$$B_2Qv = \frac{mv^2}{r}$$

$$v = \left[\frac{B_2Qr}{m}\right] \rightarrow (ii)$$

Equating equations (i) and (ii) gives

$$\frac{B_2Qr}{m} = \frac{E}{B_1}$$

Therefore (charge to mass ratio) = $(\frac{Q}{m}) = \frac{E}{rB_{1B_2}}$

Where m is the mass of the ion.

POINTS TO NOTE:

From $\left(\frac{Q}{m}\right) = \frac{E}{rB_1B_2}$

$$r = \frac{mE}{B_1B_2Q} \rightarrow (iii)$$

Therefore if B_1, B_2 and E are constants, then r∝ m hence isotopes strike the photographic plate at different points.

Point 1: If x is the distance between the slit S and the point on the photographic plate where the ion of mass m strikes,

$$\boldsymbol{r = \frac{x}{2} \rightarrow (iv)}$$

Equating equations (iii) and (iv) gives

$$\boldsymbol{\frac{x}{2} = \frac{mE}{B_1B_2Q}}$$

$$x = \boldsymbol{\frac{2mE}{B_1B_2Q}}$$

Therefore x = [$\boldsymbol{\frac{2E}{B_1B_2Q}}$]

This implies that for two ions of mass m_1 and m_2 whose distances between the slit S_3 and the point where they strike the photographic plate are x_1 and x_2 respectively, then

$$x_1 - x_2 = \left[\frac{2E}{B_1B_2Q}\right]m_1 - \left[\frac{2E}{B_1B_2Q}\right]m_2$$

$$x_1 - x_2 = \left[\frac{2E}{B_1B_2Q}\right](m_1 - m_2)$$

Since E, B_1, B_2 and Q are constant for a given spectrometer, then

$$(x_1 - x_2) \propto (m_1 - m_2) \rightarrow (v)$$

Point 2: If t is the time taken to move a horizontal distance *x,* let time taken to make a deflection of 360° be *T.* It implies that For a deflection of 360°,

$$[time\ taken] = [period]$$

$$T = \frac{2\pi}{\omega}$$

For a deflection of 180°,

$$[time\ taken] = \frac{1}{2}[period]$$

$$t = \frac{1}{2}\left[\frac{2\pi}{\omega}\right] \quad = \frac{\pi}{\omega} \quad \rightarrow (vi)$$

Also, if *v* is the angular velocity of the ion as it moves from the slit S_3 to the photographic plate, then

$$v = r\omega \quad \rightarrow (vii)$$

Equating equations *(ii)and (vii)* gives

$$r\omega = \frac{B_2Qr}{m}$$

$$\omega = \frac{B_2Q}{m}$$

Therefore $t = \frac{\pi}{\omega} \quad = \frac{\pi}{\left(\frac{B_2Q}{m}\right)} \quad = \frac{\pi m}{B_2Q}$

It therefore implies that; $t_1 = \left(\frac{\pi}{B_2Q}\right)m_1$ and $t_2 = \left(\frac{\pi}{B_2Q}\right)$

And $(t_1 - t_2) = \left(\frac{\pi}{B_2Q}\right)(m_1 - m_2)$

Thus, $(t_1 - t_2) \propto (m_1 - m_2)$

Point 3: if both ions have the same velocity that is $v_1 = v_2 = v$.

Then from equation (ii)

$$r_1 = \frac{m_1v}{B_2Q} \quad \text{and} \quad r_2 = \frac{m_2v}{B_2Q}$$

Therefore, the average radius is given by

$$r = \frac{1}{2}(r_1 - r_2) \qquad = \frac{v}{B_2 Q}(m_1 - m_2)$$

WORKED EXAMPLES

1. A charged oil drop falls under gravity with a velocity of 1.44 x 10^{-4} m/s between two horizontal metal plates 1.5cm apart. When a p.d of 5600V is applied between the plates, the drop remains

stationery. Calculate the

i. Radius of the drop,

ii. Charge on the drop.

[Density of oil = 800 kgm^{-3}, Coefficient of viscosity of air = $1.8 \times 10^{-5} Nsm^{-2}$]

Solution

v_0 = 1.44 x $10^{-4}ms^{-1}$, d = 1.5cm = 0.015m, V = 5600V, $\rho = 800kgm^{-3}, \eta = 1.8 \times 10^{-5} Nsm^{-2}$

$$r = \sqrt{\frac{9\eta v_0}{2\rho g}} = \sqrt{\frac{9 \times 1.8 \times 10^{-5} \times 1.44 \times 10^{-4}}{2 \times 800 \times 9.81}} = 1.22 \times 10^{-6} m$$

$$E = \frac{V}{d} = \frac{5600}{0.015} = 3.733 \times 10^5 NC^{-1}$$

$$EQ = \frac{4}{3}\pi r^3 \rho g$$

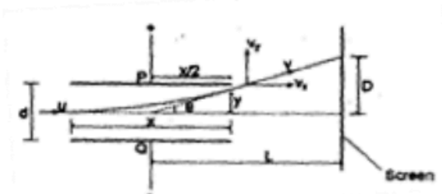
$$(3.733 \times 10^5)Q = \frac{4}{3}\pi \times (1.22 \times 10^{-6})^3 \times 9.81$$

$$(3.733 \times 10^5)Q = 5.96 \times 10^{-14}$$

$$Q = \frac{5.96 \times 10^{-14}}{3.733 \times 10^5} = 1.597 \times 10^{-19} C$$

2. An electron of energy 10keV enters mid-way between two horizontal plates each of length 5cm and separated by a distance of 2cm. A p.d of 20V is supplied across the plates. A fluorescent screen is placed 20cm beyond the plates. Calculate the vertical deflection of the electron on the screen.

Solution



$x = 5cm = 0.05m, d = 2cm = 0.02m, V = 20V$

$L = 20 + \frac{5}{2} = 22.5cm = 0.225m$

$K.E = 10keV = 10000 \times 1.6 \times 10^{-19} = 1.6 \times 10^{-15}J$

Kinetic energy

$$K.E = \frac{1}{2}mu^2$$

$$1.6 \times 10^{-15} = 0.5 \times 9.11 \times 10^{-31} \times u^2$$

$$u = \sqrt{\frac{1.6\times10^{-15}}{0.5\times9.11\times10^{-31}}} = 5.93 \times 10^7 ms^{-1}$$

Electric field intensity is given by;

$$E = \frac{V}{d} = \frac{20}{0.02} = 1000V$$

Therefore, the deflection is given by;

$$D = \left(\frac{e}{m}\right)\left(\frac{x}{u^2}\right)EL$$

$$D = \left(\frac{1.6\times10^{-19}}{9.11\times10^{-31}}\right)\left[\frac{0.05}{5.93\times10^7}\right](1000 \times 0.225)$$

$$= 5.62 \times 10^{-4}m$$

3. A cathode ray oscilloscope has its Y-sensitivity set to. A sinusoidal input voltage is applied to give a steady trace so that the electron beam takes 0.01s to transverse the screen. If the trace screen has a peak-to-peak height of 4cm and contains two complete cycles, find;

i. The root mean square value of the input voltage.

ii. The frequency of the input signal.

Solution

$$\begin{pmatrix} peak\ to\ peak \\ value\ of\ voltage \end{pmatrix} = \begin{pmatrix} peak\ to\ peak \\ length, L_0 \end{pmatrix} \times (Y - sensitivity)$$

$$= 4 \times 10 = 40V$$

$$(peak\ a.c\ voltage) = \frac{1}{2}\begin{pmatrix} peak\ to\ peak \\ value\ of\ voltage \end{pmatrix}$$

$$V_{peak} = \frac{1}{2} \times 40 = 20V$$

Root mean square value of the input voltage

$$V_{rms} = \frac{V_{peak}}{\sqrt{2}} = \frac{20}{\sqrt{2}} = 14.14\ V$$

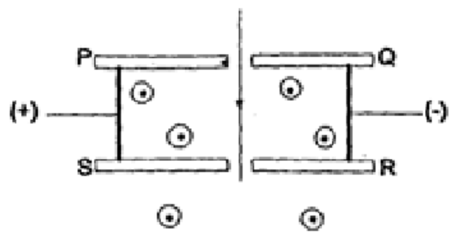
(ii)

$$2T = 0.01$$

$T = 0.005s$

$f = \frac{1}{T} = \frac{1}{0.005} = 200Hz$

4. Positive ions of the same charge are directed through slit S_a into region PQRS as shown below. There is a uniform electric field intensity of 300NC^{-1} between the plates PQ and RS. A uniform magnetic field of flux density 0.6T is directed perpendicularly out of the plate SR as shown.



(i) Calculate the velocity of the ions which go through slit S_2

(ii) Determine the motion of the ions in the region below SR

Solution

(i) Magnetic force = Electric force

$BQv = EQ$

$v = \frac{E}{B} = \frac{300}{0.6} = 500ms^{-1}$

(ii) Below SR, only magnetic field exists. The path of the ions is therefore circular and perpendicular to the magnetic field due to the magnetic force producing a centripetal force.

Trial questions

1. A charged oil drop of radius $7.26 \times 10^{-7}m$ and density of 880kgm^{-3} is held stationary in an electric field of intensity $1.7 \times 10^4 Vm^{-1}$. How many electronic charges are in the drop given the density of air is 1.29kgm^{-3} [Ans: n = 5]
2. Oil drops are introduced into the space between two flat parallel horizontal plates 5mm apart. The plate voltage is adjusted to 780V so that one drop is held stationary. The plate voltage is switched off and the selected drop is observed to fall through a distance of 1.5mm in a time of 11.2s given the density of oil is 900kgm^{-3} and coefficient of viscosity of air is $1.8 \times 10^{-5}Nsm^{-2}$ Calculate the charge on the drop. [Ans: $3.23 \times 10^{-19}C$]
3. In a CRO, an electron beam passes between the y-deflector plates each of 5cm long and 0.5cm apart. The distance between the centre of the y-plate and the screen is 20cm and the p.d between the

anode and the electron gun is 250V. Determine the deflection of the electron beam on the screen of the CRO. [Ans: 1.125m]

4. A particle of charge 3.2×10^{-19}C is accelerated from rest through a p.d of 10V. It enters into a region of uniform magnetic field of flux density 0.5T. The particle describes a circular arc of radius 8.94cm. Find;

i. The kinetic energy of the particle on entering the magnetic field,

ii. The mass of the particle.

[Ans: $3.2 \times 10^{-15}J$, 3.197×10^{-26}kg]

5. Two isotopes form a beam of positive ions. Each ion carries a charge of 1.6×10^{-19}C moving with a speed of 1.58×10^{4} m/s. The ions enter perpendicularly into a region of uniform magnetic field of flux density 0.1T which deflects the ions through 180° before they strike a photographic plate. Given that the separation between the two images on the plate caused by the ions is 6.56mm and atomic mass of the higher isotope is 240U

i. Draw a sketch diagram to show the path of the ions,

ii. Find the atomic mass of the heavier isotope,

iii. Calculate the difference in the times spent by the two isotopes in the magnetic field,

iv. Find the average radius of the paths. [$1U = 1.66 \times 10^{-27}$ kg]

[Ans: 242.001U , $6.523 \times 10^{-7}s$, 0.395]

SIR ISAAC NEWTON'S BIOGRAPHY

Without Newton, Physics would be incomplete! It is therefore worthwhile to know who Newton is.

Isaac Newton's life can be divided into three quite distinct periods. The first is his boy hood days from1643 up to his appointment to a chair in 1669. The second period from 1669 to 1687 was the highly productive period in which he was a lucasian professor at Cambridge. The third period (nearly as long as the other two combined) saw Newton as a highly paid government official in London with little further interest in mathematical research.
Isaac Newton was born in the manor house of Woolsthorpe, near Grantham in Lincolnshire, and according to the calendar in use at the time of his birth, he was born on Christmas day 1642, Isaac came from a family of farmers but never knew his father, also named Isaac Newton, who died in October 1642, three months before his son was born.

Isaac's mother ***Hannah Ayscough*** *remarried* ***Barnabas Smith*** *the minister of the church at North Witham, a nearby village, when Isaac was two years old. The young child was then left in the care of his grandmother Margery Ayscough at Woolsthorpe.*
Upon the death of his stepfather in 1653, Newton lived in an extended family consisting of mother, his grandmother, one half-brother, and two half-sisters. From shortly after this time Isaac began attending the Free Grammar school in Grantham. However he seems to have shown little promise in academic work. His school reports described him as 'idle' and 'inattentive'. His mother, by now a lady of reasonable wealth and property, thought that that her eldest son was the right person to manage her affairs and her state.
An uncle, ***William Ayscough****, decided that Isaac should prepare for entering university and, having persuaded his mother that this was the right thing to do, Isaac was allowed to return to Free Grammar School in Grantham, in 1660 to complete his school education. This time he lodged with Stokes, who was the headmaster of the school, and it would appear that, despite suggestions that he had previously shown no academic promise, Isaac must have convinced some of those around him that he had academic promise.*
Newton entered his uncle's old College, Trinity College Cambridge on 5 June 1661. He was older than most of his fellow students but, despite the fact that his mother was well off, he entered as a sizar. A ***sizar*** *at Cambridge was a student who received an allowance towards college expenses in exchange for acting as a servant to other students.*
Newton's aim at Cambridge was a law degree. Instruction at Cambridge was dominated by the philosophy of Aristotle but some freedom of study was allowed in the third year of the course. Newton studied the philosophy of Descartes, Gassendi, Hobbes, and in particular ***Boyle****. The mechanics of the Copernican astronomy of Galileo attracted him and he also studied* ***Kepler's*** *optics.*

How Newton was introduced to the most advanced texts of his day is slightly less clear. According to ***de-Moivre****, Newton's interest in mathematics began in the autumn of 1663 when he bought an*

astrology book at a fair in Cambridge and found that he could not understand the mathematics in it.
Despite some evidence that his progress had not been particularly good, Newton was elected a scholar on 28 April 1664 and received his bachelor's degree in April 1665. It would appear that his scientific genius had still had still not emerged, but it did so suddenly when the plague closed the university in summer of 1665, and he had to return to Lincolnshire. There, in a period of less than two years, while Newton was still under 25 years old, he began revolutionary advances in Mathematics, optics, physics and astronomy. In particular, during that time, he invented calculus, discovered the law of universal gravitation, proved experiments that white light is composed of seven colours.

Newton's first work as a Lucasian professor was on optics and this was the topic of his first lecture course begun in January 1670. He argued that white light is really a mixture of many different types of rays which are refracted at slightly different angles, and that each different type of ray produces a different spectral colour. Newton was led by this reasoning to the erroneous conclusion that telescopes using refractive lenses would always suffer chromatic aberration. He therefore proposed and constructed a ***reflecting telescope****.*

In 1672, Newton was elected a fellow of the Royal Society after donating a reflecting telescope. Also in 1672, Newton published his first scientific paper on light and colour in the Philosophical transactions of the Royal Society. The paper was generally well received but ***Hooke and Huygens*** *objected to Newton's attempt to prove, by experiment alone, that light consists of the motion of small particles rather than waves.*

Newton's relations with Hooke deteriorated further when, in 1675, Hooke claimed that Newton had stolen some of his optical results. Although the two men made their peace with an exchange of polite letters, Newton turned in on himself and away from the ***Royal Society*** *which he associated with Hooke as one of its leaders.*

Another argument, this time with the English Jesuits in Liege over his theory of colour, led to a violent exchange of letters, then in 1678. Newton appears to have suffered a nervous breakdown. His mother died in the following year and he withdrew further into his shell, mixing as little as possible with people for a number of years.

After suffering a second nervous breakdown in 1693, Newton retired from research. The reasons for this breakdown have been discussed by biographers and many theories have been proposed: chemical poisoning as a result of alchemy experiments, frustrations with his researches, the ending of a personal friendship with ***Fatio de Duillier****, a Swiss born mathematician resident in London, and problems resulting from his religious beliefs. Newton himself blamed lack of sleep but this was almost certainly a symptom of the illness rather than the cause of it. There seems little reason to suppose that the illness was anything other than the depression, a mental illness he must suffered from throughout most of his life, perhaps made worse by some of the events we have just listed.*
Newton decided to leave Cambridge to take up a government position in London becoming Warden of the royal Mint in 1696 and master in 1699. However, he did not resign his positions at

Cambridge until 1701. As Master of the Mint, adding the income from his estates, we see that Newton became a very rich man. For many people a position such as Master of the Mint would have been treated as simply a reward for their scientific achievements. Newton did not treat it as such and he made a strong contribution to the work of the Mint. He led it through the difficult period of recoinage and he was particularly active in measures to prevent counterfeiting of the coinage.

In 1703, he was elected president of the Royal Society and he was re-elected each year until his death. He was knighted by ***Queen Anne****, the first scientist to be so honoured for his work. However the last portion of his life was not an easy one, dominated in many ways with the controversy with* ***Leibniz*** *over which had invented the calculus.*

Given the rage that Newton had shown throughout his life when criticized, it is not surprising that he flew into an irrational temper directed against Leibniz. In 1716 Leibniz proposed what seemed to him the most difficult problem as a challenge to Europe mathematicians, but was actually aimed at Newton. Newton received this problem at about 5:00pm and solved it that very evening. He (Newton) later used his position as President of the Royal Society. In this capacity, he appointed an ''impartial'' committee to decide whether he or Leibniz was the inventor of calculus. He wrote the official report of the committee (although of course it did not appear under his name)
Isaac Newton died on 31 March 1727 at the age of 84.

In conclusion*: Newton laid the foundation for differential and integral calculus. His work on optics and gravitation make him one of the greatest scientists the world has known.*

REFERENCES

Advanced Level Physics by Roger Muncaster Fourth Edition Nelson Thornes

Advanced Level Physics by Nelkon & Parker Seventh Edition Heinemann

Advanced Level Physics by Tom Duncan Fifth Edition

Simplified Advanced Level Physics, Mechanics and Properties of Matter by Mukose Sowali